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Wind-wave generation by energy and momentum flux to the forced components of a wave field

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The response of surface gravity waves with profile ζ to forcing by the atmospheric pressure $p^{(a)}$ is studied. It is shown that not only the usually considered cross-spectrum $\langle p^{(a)}\zeta\rangle$ is responsible for wave growth. All higher-order spectra $\langle p^{(a)}\zeta_1, \ldots, \zeta_n\rangle$ also lead to wave growth linear in $\epsilon = \rho^{(a)}/\rho$, $\rho^{(a)}$ and ρ being the density of air and water respectively. The bispectral contribution is investigated in detail and is calculated for three models of the bispectrum. The results indicate that bispectral contributions are not negligible and may account for typically 30–50 %, and possibly more, of the usually considered cross-spectral contribution. The bispectral growth mechanism contains the fluctuating-stress mechanism discussed by Longuet-Higgins (1969) as a special case. As a byproduct new information on the symmetry properties of the hydrodynamic coupling off the dispersion shell is obtained.

1. Introduction

1.1. *The JONSWAP balance and the resolution of an apparent paradox*

The understanding of wave generation has made considerable progress over recent years. The JONSWAP measurements (Hasselmann *et al.* 1973) led to a balance between nonlinear wave–wave interaction, atmospheric input and dissipation. Under the assumption that the atmospheric input and dissipation are essentially local in wavenumber space, so that they do not transfer energy across the spectrum, a minimum wave-induced stress described by a minimum wave drag coefficient C_w can be derived:

$$\tau_{\min} = C_w U_{10}^2. \tag{1.1}$$

With some scatter the atmospheric-input measurements of Dobson (1971), Elliott (1972) and Snyder (1974) can be described by

$$DF(\sigma)/Dt = \mu\sigma(U/c-1)F(\sigma) \tag{1.2}$$

with μ varying from experiment to experiment ($\mu = 1{\cdot}0$ for Dobson's measurements, $\mu = 0{\cdot}2$ for Elliott's and $\mu = 0{\cdot}1$ for Snyder's). Here U is the 5 m wind speed, $F(\sigma)$ is the wave spectrum and $c = g/\sigma$ is the phase speed.

Applying (1.2) to a mean JONSWAP spectrum, defined by five parameters, Snyder finds that all measured inputs satisfy the minimum input requirements (Snyder 1974, figure 19). If, however, we impose the condition that the minimum input must be supplied in the range $0 \leqslant \sigma \leqslant \sigma_{hf}$, with $\sigma_{hf} \approx \frac{7}{3}\sigma_{\max}$ and $\sigma_{\max}$ the frequency at the spectral peak, we find that Snyder's measurements can no longer supply the minimum drag. (We have done the same calculations as Snyder, except that we have replaced his value of $I_2 = 0{\cdot}88$ by $\hat{I}_2 = 0{\cdot}88 - \frac{3}{7}$ and his value of I_3 by $\hat{I}_3 = 0{\cdot}51 - \frac{1}{2}(\frac{3}{7})^2$. A similar calculation with a $\delta(\Theta)$ instead of a $\cos^2\Theta$ spread function does not materially alter this conclusion, although a $\delta(\Theta)$ spread yields a higher input. In the latter case Snyder's measurements can supply typically 80 % of the minimum stress.)

Certainly (1.2) is only a crude parameterization and wave spectra also scatter around the average JONSWAP spectrum. Nevertheless it seems worth while to reconsider the assumption that dissipation and atmospheric input are essentially local processes in wavenumber space. We shall here be concerned with atmospheric input only.

Disregarding experimental error, the discrepancies among the measurements of the atmospheric source function mentioned after (1.2) can be explained only by assuming that the wave-induced pressure is a nonlinear function of wave height. If nonlinear terms are important for the transfer of energy and momentum to the free waves of the sea, why should they not be able to transfer energy and momentum to the forced surface waves? (Free waves are the Fourier components that lie on the dispersion shell for linear waves $\omega^2 = kg$, i.e. they lie on the resonance shell of the sea surface, i.e. they are on-shell. Forced waves are off-shell.)

As far as transfer is concerned there is no reason why the atmosphere should make a distinction between free and forced surface waves. The only reason why the transfer to forced waves might be negligible is that forced waves are smaller than free waves. (Sometimes the argument is heard that transfer to forced waves must be quadratic in $\epsilon = \rho^{(a)}/\rho$, or even that there can be no transfer to forced waves, because these are not in resonance. Such arguments must be wrong, since the atmosphere is free to take advantage of any surface elevation; it does not care about the dynamics of the sea surface. Even if the atmosphere did care, we could still construct a machine that would generate surface pressure in a way we prescribed.)

As far as wave growth is concerned the situation is simple for free waves: they undergo resonant forcing, they grow and input equals growth.

For forced waves a paradox seems to arise: although there may be input, we do not see any growth. The waves are hydrodynamically forced by free waves, thus not resonantly, and additional out-of-phase atmospheric forcing will neither cause growth nor shift the phase sufficiently to terminate the input. There is only one possibility: the energy and momentum transferred to the forced waves must reappear in the wave field as growth of the forcing components. (Of course, this is a theoretical argument, but no assumptions regarding wave dynamics are made other than the assumption that for small waves an expansion in wave slope is feasible and the assumption that forcing by atmospheric pressure is small compared with nonlinear hydrodynamic forcing.)

This mechanism and an estimate of its effectiveness constitute the subject of this paper. In the appendix we discuss the symmetry properties of the hydrodynamic coupling. Dissipation processes are not discussed, but we mention that they could show a similar spectral behaviour to the mechanism discussed here.

1.2. *An introductory example*

The mechanism of energy and momentum transfer to forced waves is very simple as can be seen from the following analogy, which contains most of the relevant physics.

Consider a conservative system of three weakly coupled oscillators, e.g. three LC-circuits weakly coupled by nonlinearities, governed by

$$\dot a_1 + i\sigma_1 a_1 = -iDa_2 a_3, \tag{1.3a}$$

$$\dot a_2 + i\sigma_2 a_2 = -iDa_1 a_3^*, \tag{1.3b}$$

$$\dot a_3 + i\sigma_3 a_3 = -iDa_1 a_2^*, \tag{1.3c}$$

where the a_i^* are governed by the complex-conjugate equations, D is real and the a_i are non-dimensional quantities $O(1)$. The Hamiltonian of this system is

$$H = \sum_{i=1}^{3} \sigma_i a_i a_i^* + Da_1^* a_2 a_3 + Da_1 a_2^* a_3^* = H_0 + H_D \tag{1.4}$$

and is conserved since

$$\dot a_i = -i\,\partial H/\partial a_i^*, \quad \dot a_i^* = i\,\partial H/\partial a_i. \tag{1.5}$$

Suppose further that the coupling D is weak, i.e. $D \ll \sigma_i$, and that the frequencies are completely mismatched, i.e.

$$\Sigma \nu_i \sigma_i \gg D, \quad \nu_i = \pm 1. \tag{1.6}$$

If the initial conditions at $t = 0$ are $(a_1, a_2, a_3) = (\alpha_1, \alpha_2, 0)$ the system will behave like $(\alpha_1 \exp(-i\sigma_1 t), \alpha_2 \exp(-i\sigma_2 t), 0)$. At the next order we obtain

$$a_3 = a_3^D = \alpha_1 \alpha_2^* \frac{D}{\sigma_1 - \sigma_2 - \sigma_3} \{\exp[-i(\sigma_1 - \sigma_2)t] - \exp(-i\sigma_3 t)\}, \quad a_3^D \ll 1. \tag{1.7}$$

At the next highest order we find a detuning of σ_1 and σ_2 corresponding to the finite energy as a_3 and such effects may eventually at some very high order allow resonance, corresponding to an extremely slow distribution of energy, from the first two modes to the third.

To fourth order, however, the mean energy in each mode, averaged over times $\sigma_i T \gg 1$, remains steady and is concentrated in the first two modes. The interaction energy H_D and the energy in the third mode are small, $O((D/\sigma)^2)\,H_0$, and are also both steady.

It is intuitively obvious that, if we draw energy out of a_3 by adding a resistance to the circuit or if we add energy by inserting a negative resistance, then a_1 and a_2 must correspondingly decay or grow. Furthermore, the energy source or sink may be quite generally due to an additional forcing term $r(t)$ in (1.3c). Thus, if now

$$\dot a_3 + i\sigma_3 a_3 = -iDa_1 a_2^* + \Sigma r_\omega e^{-i\omega t}, \quad r_\omega \quad \text{real}, \tag{1.8}$$

the solution is

$$a_3 = a_3^D + a_3^r$$

with

$$a_3^r = \Sigma r_\omega [\exp(-i\omega t) - \exp(-i\sigma_3 t)]/i(\sigma_3 - \omega). \tag{1.9}$$

For simplicity, we set $r_\omega = 0$ at $\omega = \sigma_3$.

If $r_\omega \neq 0$ for $\omega = \sigma_1 - \sigma_2$ then the time average $\overline{\dot a_3 r} = \overline{\dot a_3^D r_\omega} \neq 0$, and from the analogy with a resistance, energy is being supplied to the system, the sign and magnitude depending on the phase of $(\alpha_1^*, \alpha_2, r_\omega)$.

In a more formal fashion, (1.5) still holds, but now

$$H = H(t) = H_0 + H_D + H_r, \quad H_r = i(ra_3^* - r^* a_3) \tag{1.10}$$

and

$$dH/dt = \partial H/\partial t = i(\dot r a_3^* - \dot r^* a_3) \neq 0. \tag{1.11}$$

Since our solution shows H_r to be steady (at least at order rD) it follows that

$$d(H_0+H_D)/dt = O(rD) \neq 0.$$

It is immediately evident that H_D is also steady at this order, so that it follows that

$$\frac{\partial H}{\partial t} = i(\dot{r}a_3^* - \dot{r}^* a_3) = \frac{d}{dt}H_0 = \frac{d}{dt}[\sigma_1(a_1 a_1^*) + \sigma_2(a_2 a_2^*)]. \tag{1.12}$$

This result follows directly from (1.3) and (1.8). Retaining only the relevant terms gives

$$\left.\begin{aligned} \dot{a}_1 + i\sigma_1 a_1 &= -iDa_2 a_3^r, \\ \dot{a}_2 + i\sigma_2 a_2 &= -iDa_1 a_3^{r*}. \end{aligned}\right\} \tag{1.13}$$

Only the terms oscillating in resonance yield unsteady solutions, since

$$\sigma_1(\dot{a}_1 a_1^* + \dot{a}_1^* a_1) = 2\sigma_1 \operatorname{Re}\{-iDa_1^* a_2 a_3^r\} = \sigma_1 A, \tag{1.14}$$

$$\sigma_2(\dot{a}_2 a_2^* + \dot{a}_2^* a_2) = -2\sigma_2 \operatorname{Re}\{-iDa_1^* a_2 a_3^r\} = -\sigma_2 A \tag{1.15}$$

and further from (1.7)–(1.9)

$$\partial H/\partial t = i(a\dot{r}_3^{*D} - \dot{r}^* a_3^D) = (\sigma_1 - \sigma_2)A, \tag{1.16}$$

so that indeed at order rD

$$\frac{dH}{dt} = \frac{dH_0}{dt} = \frac{\partial H}{\partial t}. \tag{1.17}$$

Furthermore, to ease the comparison with results given later, we note that

$$A = 2\operatorname{Re}\{\alpha_1 \alpha_2^* r_{\omega'} D/(\omega' - \sigma_3)\}, \quad \omega' = \sigma_1 - \sigma_2. \tag{1.18}$$

The interaction we have considered in this example is called a difference interaction, while an interaction described by $H_D^{(+)} = D\{a_1 a_2 a_3^* + a_1^* a_2^* a_3\}$ is called a sum interaction. For the difference interaction we have conservation of action, i.e.

$$\frac{d}{dt}(a_1 a_1^*) + \frac{d}{dt}(a_2 a_2^*) = 0, \tag{1.19}$$

so that (for $\sigma_1 - \sigma_2 > 0$) an increase of energy H_0 (positive energy flux $\partial H/\partial t$) is associated with an increase of energy and action in the first mode and a decrease in the second. For the sum interaction we have an increase of action

$$\frac{d}{dt}\{a_1 a_1^* + a_2 a_2^*\} = 2A = 2\frac{d}{dt}(a_1 a_1^*) = 2\frac{d}{dt}(a_2 a_2^*) \tag{1.20}$$

and in this case a positive energy flux is associated with an increase of energy and action in both modes. The same behaviour is found later in the wave case, and we see that the differences in behaviour between sum and difference interactions have their root in the action balance.

While the extension to the statistical case is not quite trivial for scattering processes (to use K. Hasselmann's notation), for our case, which involves parametric forcing only, the passage to statistics only involves placing angle brackets around steady terms.

The major differences between this example and the problem of interacting surface waves are as follows.

(i) We shall have a continuum of modes $a(k)$. This presents no obstacles.

(ii) We have a large range of amplitudes $a(k)$, i.e. large waves at small k and small waves at large k. This results in a coupling between long waves and short waves, which is strong for the short waves. This presents a problem which, however, will be almost completely ignored in this paper, except for a brief discussion in §4.2. In a forthcoming paper we show that weak interaction theory can handle the case of strong coupling in a deterministic situation.

(iii) The equations of motion for surface waves do not have the canonical form (1.3) but also contain terms of the type

$$\dot{a}_1 + i\sigma_1 a_1 = -iD\dot{a}_2 a_3 + \ldots. \tag{1.21}$$

This feature, combined with the fact that symmetry properties are not so transparent in the a, a^* representation, has led to a formulation in terms of the second-order equation

$$\ddot{z}_1 + \sigma_1^2 z_1 = D_1 \ddot{z}_2 z_3 + D_2 \dot{z}_2 \dot{z}_3 + \ldots, \tag{1.22}$$

but no changes in the principal mechanism discussed above result.

2. Formulation and mathematical development

2.1. *Notation and preliminaries*

We consider an inviscid homogeneous ocean of infinite depth, use the co-ordinates (x, y, z) or $(1, 2, 3)$ interchangeably and let the mean water surface be at $z = 0$. Then $z = \zeta(x, y, t)$, say, will describe the water surface and $p(z = \zeta) = p^{(a)}(x, y, t)$, say, will be the surface pressure acting on it. (All motion is considered to be irrotational.) Finally, ρ and $\rho^{(a)}$ denote the density of air and water. We decompose ζ into spatial Fourier components

$$\zeta = \sum_{\mathbf{k}} \zeta_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.1}$$

by Fourier integration over a large area L^2, which we can allow to go to infinity in the final results.

Statistical averages will be defined as ensemble averages (denoted by angle brackets), with the understanding that sufficiently, but not too widely, separated space–time domains will define an ensemble, without going into any of the problems associated with such an interpretation.

The equations of motion for $\zeta_{\mathbf{k}}$ are then

$$\ddot{\zeta}_{\mathbf{k}} + \sigma_k^2 \zeta_{\mathbf{k}} = -(\sigma_k^2/\rho g)\, p_{\mathbf{k}}^{(a)} + \sigma_k^2 N_{\mathbf{k}}. \tag{2.2}$$

Here $\sigma_k = (kg)^{\frac{1}{2}} > 0$ $(k = |k|)$ is the deep-water wave frequency and $N_{\mathbf{k}}$ is an abbreviation for all hydrodynamic interaction terms nonlinear in the wave height. Thus (2.2) represents a system of weakly coupled oscillators and we shall here mainly be interested in the influence of weak cross-coupling due to the influence of $p_{\mathbf{k}}^{(a)}$ in combination with the hydrodynamic coupling $N_{\mathbf{k}}$.

We point out that $N_{\mathbf{k}}$ is a time-dependent operator containing terms $\partial^2(\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\ldots)/\partial t^2$, which makes it advisable to consider (2.2) in the frequency domain. With

$$\omega = \omega_0 + i\lambda \quad (\lambda > 0),$$

$$Z(\mathbf{k}, \omega) = \frac{1}{2\pi}\int_0^\infty e^{i\omega t}\,\zeta_{\mathbf{k}}(t)\,d\omega, \tag{2.3}$$

$$b_{\mathbf{k}}(t) = p_{\mathbf{k}}^{(a)}/\rho^{(a)}$$

and

$$b(\mathbf{k}, \omega) = \frac{1}{2\pi}\int_0^\infty e^{i\omega t}\,b_{\mathbf{k}}(t)\,d\omega, \tag{2.4}$$

we obtain

$$(-\omega^2 + \sigma^2)\,Z(\mathbf{k}, \omega) = -\frac{\rho^{(a)}}{\rho}\frac{\sigma^2}{g}\,b(\mathbf{k}, \omega) + \sigma^2 N(\mathbf{k}, \omega) + d(\mathbf{k}, \omega) \tag{2.5}$$

with $d(\mathbf{k}, \omega)$ determined from the initial conditions. The continuation of $Z(\mathbf{k}, \omega)$ from values of ω in the upper half-plane to values of ω on the real axis then determines the time development of $\zeta_{\mathbf{k}}$:

$$\zeta_{\mathbf{k}}(t) = \int_{-\infty}^{+\infty} e^{-i\omega t}\,Z(\mathbf{k}, \omega)\,d\omega. \tag{2.6}$$

The statistics for the forcing terms of (2.5), given by

$$C_{nm} = \langle b_i \ldots b_n d_j \ldots d_m\rangle, \quad m, n = 0, \ldots, \infty \tag{2.7}$$

(here the indices b_i and d_j stand for $b(\mathbf{k}_i, \omega_i)$ and $d(\mathbf{k}_j, \omega_j)$), then determine the spectra

$$F_{\zeta\zeta}(\mathbf{k}, t) = \langle \zeta_{\mathbf{k}}(t)\,\zeta_{-\mathbf{k}}(t)\rangle/\Delta k\,.\,\Delta k \tag{2.8}$$

and

$$F_{b\zeta}(\mathbf{k}, t) = \langle b_{\mathbf{k}}(t)\,\zeta_{-\mathbf{k}}(t)\rangle/\Delta k\,.\,\Delta k. \tag{2.9}$$

From these we can obtain the growth rates and input fluxes. (It is clear that the method of ensemble averaging, with ensembles defined as above, introduces certain weak constraints on the wavenumber–frequency structure of the forcing statistics C_{nm}. Essentially this means that $\delta(\mathbf{k} - \mathbf{k}_1)\,\delta(\omega - \omega_1)$ amplitudes must not develop over our averaging times and lengths. If for instance the atmosphere had an extremely sharp isolated resonance at $k = 0{\cdot}1\,\mathrm{m}^{-1}$ and $\omega = 0{\cdot}5\,\mathrm{s}^{-1}$ this would show up as a small deterministic signal in the surface wave field (even if at $t = 0$ the wave field had no such signal), producing a significant coherence between different samples out of our ensemble.)

Returning to (2.5) we note that the nonlinear coupling term N has the form

$$N\{Z\}(\mathbf{k}, \omega) = \sum_{n=3}^{\infty}\int D_n(\mathbf{k}_1, \omega_1; \ldots; \mathbf{k}_n, \omega_n)\,Z(\mathbf{k}_2, \omega_2)\ldots Z(\mathbf{k}_n, \omega_n)$$
$$\times\,\delta(\Sigma\mathbf{k}i)\,\delta(\Sigma\omega i)\,\delta(\mathbf{k}_1 + \mathbf{k})\,\delta(\omega_1 + \omega)\,d\omega_1\ldots d\omega_n\,d\mathbf{k}_1\ldots d\mathbf{k}_n. \tag{2.10}$$

In the appendix [equations (A 25)–(A 29)] it is shown how to construct the D_n and that these are symmetric; thus with $\alpha_i = (\mathbf{k}_i, \omega_i)$

$$D_n(\alpha_1, \ldots, \alpha_n) = D_n(\alpha_{\nu_1}, \ldots, \alpha_{\nu_n}) \tag{2.11}$$

with $\nu_1, \ldots, \nu_n$ any permutation of $i, \ldots, n$. The relation between these coupling functions D_n and the frequency-independent coupling functions $D(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ introduced by Hasselmann (1966a, b) is also discussed in the appendix.

For completeness we list the three-point coupling coefficients (for deep water), since we shall need these later:

$$D(\alpha_1, \alpha_2, \alpha_3) = (2g)^{-1}\{\omega_1^2+\omega_2^2+\omega_3^2-2\omega_1\omega_3\cos(1,3) \\ -2\omega_2\omega_3\cos(2,3)-2\omega_1\omega_2\cos(1,2)\}, \quad (2.12)$$

where $\cos(i,j) = \cos(\mathbf{k}_i, \mathbf{k}_j)$. Thus (2.5) is an integral equation for $Z(\mathbf{k}, \omega)$ and for sufficiently weak coupling an iterative solution can be found with the help of Feynman diagrams. The method is well known, but for completeness a brief description is given in the appendix.

The growth rate of the wave spectrum at the wavenumber $\mathbf{k}$ is given by

$$\frac{d}{dt}\langle \zeta_{\mathbf{k}}(t)\,\zeta_{-\mathbf{k}}(t)\rangle = \int \Omega\, e^{-i\Omega t}T(\mathbf{k}, \Omega)\, d\Omega, \quad (2.13)$$

with

$$T(\mathbf{k}, \Omega) = \operatorname{Im}\int \langle Z(\mathbf{k}, \Omega+\omega)\, Z(-\mathbf{k}, -\omega)\rangle\, d\omega. \quad (2.14)$$

Thus, if $T(\mathbf{k}, \Omega)$ has a double pole at $\Omega = 0$, $dF_\zeta(\mathbf{k})/dt$ will have a steady mean. Similarly the input fluxes of energy and momentum will have a steady mean if $G(\mathbf{k}_i, \Omega)$, defined by

$$G(\mathbf{k}, \Omega) = \operatorname{Im}\int_{-\infty}^{+\infty} b(\mathbf{k}, \Omega+\omega)\, Z(-\mathbf{k}, -\omega)\, d\omega, \quad (2.15)$$

has a pole at $\Omega = 0$. (See also equations (A 7) and (A 8) in the appendix.)

2.2. *The bispectral contributions to the input fluxes and growth rates*

The bispectrum $B(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2)$ of pressure and two wave components is defined by

$$\langle b(\mathbf{k}, \omega)\, Z(\mathbf{k}_1, \omega_1)\, Z(\mathbf{k}_2, \omega_2)\rangle = B(\mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2)\, \delta(\omega_1^2-\sigma_1^2) \\ \times \delta(\omega_2^2-\sigma_2^2)\, \delta(\omega_1+\omega_2+\omega_3)\, \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \quad (2.16)$$

which describes the limit for large averaging times and large averaging areas of averages obtained from standard (fast Fourier transform) time-series analysis.

If

$$\hat{a}(\mathbf{k}_n, \omega_n) = \frac{1}{L^2T}\int_{L^2T} a(\mathbf{x}, t)\exp[-i(\mathbf{k}_n x - \omega_n t)]\, dt\, dx, \quad (2.17)$$

then

$$a(\mathbf{k}, \omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} a(x, t)\exp[-i(\mathbf{k}x - \omega t)]\, dx\, dt \quad (2.18)$$

is the (only, in a generalized sense, existent) limit

$$a(\mathbf{k}, \omega) = \lim_{\Delta k,\, \Delta\omega\to 0}\{\hat{a}(\mathbf{k}_n, \omega_n)/(\Delta k)^2\Delta\omega \quad (2.19)$$

and correspondingly we can define $B_{lmn} = B(\mathbf{k}_1, \omega_1;\, \mathbf{k}_m;\, \omega_m;\, \mathbf{k}_n, \omega_n)$ by

$$(\Delta\omega)^{-3}(\Delta k)^{-6}\langle \hat{b}_l \hat{Z}_m \hat{Z}_n\rangle = B_{lmn}\,\tilde{\delta}(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n) \\ \times \tilde{\delta}(\omega_l+\omega_m+\omega_n)\, \tilde{\delta}(\omega_m^2-\sigma_m^2)\, \tilde{\delta}(\omega_n^2-\sigma_n^2). \quad (2.20)$$

Here the $\tilde{\delta}$'s are discrete approximations of the δ-functions, e.g.

$$\tilde{\delta}(\omega_m+\omega_n) = (\Delta\omega)^{-1}\delta_{m,-n}, \quad (2.21)$$

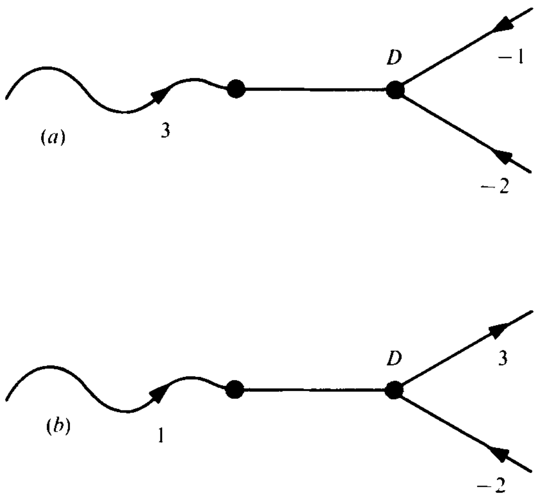

FIGURE 1. The Feynman diagrams corresponding to bispectral transfer expressions (2.28)–(2.34). The wiggly line represents the atmospheric pressure. (a) The pressure component $b(\alpha_3)$ can do work on the forced wave set up by the free waves $Z(-\alpha_1)$ and $Z(-\alpha_2)$ for $\alpha_3 = \alpha_1 + \alpha_2$. (b) The pressure component $b(\alpha_1)$ will produce a forced wave $Z(\alpha_1)$, and interaction with the free wave $Z(-\alpha_2)$ produces a free wave $Z(\alpha_3)$ ($\alpha_3 = \alpha_1 - \alpha_2$), leading to growth of the wave spectrum at α_3. (The α_i for (a) and (b) are not identical.)

where $\delta_{m,-n}$ is the Kronecker symbol. The factor

$$\tilde{\delta}(\omega_m^2 - \sigma_m^2) = (2\sigma_m)^{-1}\{\tilde{\delta}(\omega_m - \sigma_m) + \tilde{\delta}(\omega_m + \sigma_m)\}$$

arises because $a(\mathbf{k}_n, w_n)$ will have spikes at $\omega_n = \pm\sigma_n$ for frequency resolutions

$$(d\sigma/dk)\,\Delta k < \Delta\omega \ll \sigma(k). \tag{2.22}$$

Thus for practical purposes, we need only the spatial resolution and (2.20) is equivalent to

$$(\Delta k)^{-4}\langle\tilde{b}(\mathbf{k}_l)\,Z^{\nu_1}(\mathbf{k}_m)\,Z^{\nu_2}(\mathbf{k}_n)\rangle = B^{\nu_1\nu_2}_{\mathbf{k}_m\mathbf{k}_n}, \tag{2.23}$$

with $\mathbf{k}_1 = -(\mathbf{k}_m + \mathbf{k}_n)$ and $\nu = \pm 1$,

$$Z^{\nu}(\mathbf{k}, t) = \tfrac{1}{2}[\zeta_{\mathbf{k}} - (\nu/i\sigma)\,\dot{\zeta}_{\mathbf{k}}] \tag{2.24}$$

and

$$B^{\nu_1\nu_2}_{\mathbf{k}_m\mathbf{k}_n} = (4\sigma_m\sigma_n)^{-1}\,B(\mathbf{k}_l, \omega_l;\, \mathbf{k}_m, \nu_1\sigma_m;\, \mathbf{k}_n, \nu_2\sigma_n). \tag{2.25}$$

We shall here however use the representation (2.16).

The relevant Feynman diagram yielding steady input fluxes is shown in figure 1 (a). The fluxes J will be normalized such that

$$J(\mathbf{k}) = J(-\mathbf{k}) \tag{2.26}$$

and

$$J_{\text{tot}} = \int_H J(\mathbf{k})\,d\mathbf{k} = \frac{1}{2}\int_F J(\mathbf{k})\,d\mathbf{k}, \tag{2.27}$$

H and F indicating 'half-plane' and 'full plane'. Figure 1 (a) then yields

$$J_E(\mathbf{k}) = 2\rho^{(a)}\int d\omega\,\omega h_2(\mathbf{k}, \omega), \tag{2.28}$$

$$J_i(\mathbf{k}) = 2\rho^{(a)}\mathbf{k}_i\int d\omega\, h_2(\mathbf{k}, \omega). \tag{2.29}$$

Here h_2 is the imaginary part of $h(\mathbf{k},\omega)$,

$$h(\mathbf{k},\omega) = h_1(\mathbf{k},\omega)+ih_2(\mathbf{k},\omega), \tag{2.30}$$

and

$$h(\mathbf{k}_3,\omega_3) = \int D_3(\alpha_3,\alpha_1,\alpha_2)\,B(\alpha_1,\alpha_2)\,\delta(\omega_2^2-\sigma_2^2) \times \delta(\omega_1^2-\sigma_1^2)\frac{\sigma_3^2}{\sigma_3^2-(\omega_1+\omega_2)^2}\delta(\alpha_1+\alpha_2+\alpha_3)\,d\alpha_1\,d\alpha_2, \tag{2.31}$$

in which $\alpha_i = (\mathbf{k}_i,\omega_i)$. Similarly, if the wave energy and momentum are again normalized according to (2.26) and (2.27), i.e.

$$E(\mathbf{k}) = E(-\mathbf{k}), \quad E_{\text{tot}} = \frac{1}{2}\int_{\mathrm{F}} E(\mathbf{k})\,d\mathbf{k},$$

we have

$$\frac{\partial}{\partial t}E(\mathbf{k}_3) = -4\rho^{(a)}\int \omega_3 H_2(\mathbf{k}_3,\omega_3)\,d\omega_3, \tag{2.32}$$

$$\frac{\partial}{\partial t}\mathbf{M}(\mathbf{k}_3) = -4\rho^{(a)}\mathbf{k}_3\int H_2(\mathbf{k}_3,\omega_3)\,d\omega_3. \tag{2.33}$$

Here H_2 is the imaginary part of $H = H_1+iH_2$, where

$$H(\mathbf{k}_3,\omega_3) = \int d\alpha_1\,d\alpha_2\,\delta(\alpha_1+\alpha_1+\alpha_3)\,\delta(\sigma_3^2-\omega_3^2) \times \delta(\sigma_2^2-\omega_2^2)\frac{\sigma_1^2}{(\sigma_1^2-\omega_1^2)}D_3(\alpha_3,\alpha_1,\alpha_2)\,B(\alpha_2,\alpha_3). \tag{2.34}$$

The formulae for growth rates and fluxes can readily be read off the diagrams. The explanation for the factor 4 instead of the perhaps expected factor 2 in (2.33) is that in (2.34) the contribution from $D_3(\alpha_3,\alpha_2,\alpha_1)$ is suppressed.

Figure 1 (*a*) shows how the pressure b_3 can transfer energy and momentum to the wave field by interacting with the forced wave Z_3 set by up Z_1^* and Z_2^*. Figure 1 (*b*) shows that a free wave Z_3 will grow, because forced waves generated by b_1 can interact with Z_2^* to produce a wave out of phase and coherent with Z_3.

2.3. *The energy and momentum balance*

The input fluxes at forced frequencies are balanced by growth at the eigenfrequencies, as can easily be seen from (2.28)–(2.34). Suppose that the bispectrum $B(\alpha_1,\alpha_2)$ vanishes everywhere unless (α_1,α_2) is one of the following pairs:

$$(\beta_1,\beta_2), \quad (\beta_2,\beta_1), \quad (-\beta_1,-\beta_2), \quad (-\beta_2,-\beta_1).$$

Then according to (2.31) input fluxes will occur only at $\alpha_3 = \beta_3 = \beta_1+\beta_2$ and at $\alpha_3 = -\beta_3$. Thus at $\mathbf{k}_3 = \mathbf{k}_1+\mathbf{k}_2$ we have

$$\int d\omega_3\,\alpha_3 h_2(\mathbf{k}_3,\omega_3) = -2(\beta_1+\beta_2)\,A \tag{2.35}$$

with

$$A = -\operatorname{Im}\frac{1}{4\sigma_1\sigma_2}D_3(-(\beta_1+\beta_2),\beta_1,\beta_2)\,B(\beta_1,\beta_2)\frac{\sigma_3^2}{\sigma_3^2-\omega_3^2}. \tag{2.36}$$

The factor 2 arises because both $B(\beta_1,\beta_2)$ and $B(\beta_2,\beta_1)$ contribute.

On the other hand the growth rates occur at only $\alpha_3 = \pm\beta_1$ or $\alpha_3 = \pm\beta_2$. Invoking the symmetry of the coupling coefficient D, we see immediately that

$$\int \alpha_3 H_2(\mathbf{k}_1, \omega_3)\, d\omega_3 = \beta_1 A, \tag{2.37}$$

and

$$\int \alpha_3 H_2(\mathbf{k}_2, \omega_3)\, d\omega_3 = \beta_2 A. \tag{2.38}$$

Thus from (2.28), (2.29), (2.32) and (2.33) we see how the energy and momentum balance is achieved: the input at the forced wave $(\mathbf{k}_3, \omega_3)$ is balanced by growth of the forcing waves at $(\mathbf{k}_1, \sigma_1)$ and $(\mathbf{k}_2, \sigma_2)$.

3. Discussion

3.1. *The sum and difference interaction*

Amusing side results can be obtained if we separately consider the sum and difference interaction of two waves $\alpha_1 = (\mathbf{k}_1, \sigma_1)$ and $\alpha_2 = (\mathbf{k}_2, \sigma_2)$. (There is no loss of generality in choosing the frequency positive for both waves.) The sum and difference interactions will force two waves $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = \alpha_1 - \alpha_2$. The input fluxes at α_3 and α_4 are proportional to the bispectra $B(\alpha_1, \alpha_2)$ and $B(\alpha_1, -\alpha_2)$ respectively.

Now, considering the difference interaction, let us assume that the energy input at $\mathbf{k}_4$ was positive, and further that $\sigma_1 > \sigma_2$ and

$$J_E(\mathbf{k}_4) = (\sigma_1 - \sigma_2) A(\alpha_1, -\alpha_2) > 0. \tag{3.1}$$

Then we see that the rate of change of wave energy

$$E = 2\rho g \langle \zeta_{\mathbf{k}} \zeta_{-\mathbf{k}} \rangle \quad \text{at} \quad \alpha_1, \alpha_2$$

is given by

$$dE(\alpha_1)/dt = \sigma_1 A(\alpha_1, -\alpha_2) > 0 \tag{3.2}$$

whereas

$$dE(\alpha_2)/dt = -\sigma_2 A(\alpha_1, -\alpha_2) < 0. \tag{3.3}$$

Since $A(\alpha_1, -\alpha_2) = -A(\alpha_2, -\alpha_1)$ we conclude that, if the difference interaction always yields positive energy flux to the wave field, then pair for pair this interaction will damp the lower-frequency wave and stimulate the high-frequency wave. On integrating over the whole spectrum we see that for a specific frequency σ_s all waves with $\sigma_1 > \sigma_s$ will damp this wave and all waves with $\sigma_1 < \sigma_s$ will cause growth, i.e.

$$\frac{d}{dt} E(\sigma_s) = -\sigma_s \int_{\sigma_1 > \sigma_s} A(\mathbf{k}_1, \sigma_1; -\mathbf{k}_s, -\sigma_s)\, d\mathbf{k}_1 + \sigma_s \int_{\sigma_1 < \sigma_s} A(\mathbf{k}_s, \sigma_s; -\mathbf{k}_1, -\sigma_1)\, d\mathbf{k}_1. \tag{3.4}$$

Thus where the exact frequency σ_z of zero growth lies depends on the distribution of A, which primarily depends on the bispectrum and thus on the wave spectrum itself. Conversely, if the difference interaction always took energy out of the wave field the general tendency would be to stimulate low-frequency and damp high-frequency waves.

The same analysis can be applied to the sum interaction, which is, however, less exotic; since $A(\alpha_1, \alpha_2) = A(\alpha_2, \alpha_1)$ we find that positive (negative) flux due to the sum interaction results in growth (decay) of the wave field at the forcing frequencies. If we allow complicated distributions of $A(\mathbf{k}_1, \sigma_1; \mathbf{k}_2, \sigma_2)$ and $A(\mathbf{k}_1, \sigma_1; -k_2, -\sigma_2)$ then obviously the distribution of growth rates over the spectrum can become arbitrarily complicated.

3.2. *The relation to the fluctuating-stress mechanism of Longuet-Higgins* (1969)

Longuet-Higgins (1969) has shown that a fluctuating stress at the surface will generate waves if the stress is in phase with the waves. The argument is not limited to a purely viscous stress. A covariance of high-frequency pressure and high-frequency wave slopes which varies slowly in phase with a long wave should also lead to growth of the long wave. Clearly, such an effect must be included in the foregoing analysis.

Consider two waves, one slowly varying, $Z(\mathbf{k}_1, \sigma_1)$, $\alpha_1 = (\mathbf{k}_1; \sigma_1)$, and one of high frequency, $Z(\mathbf{k}_2, \sigma_2)$, $\alpha_2 = (\mathbf{k}_2; \sigma_2)$. Their interaction will create forced waves Z_3 and Z_4 at

$$\alpha_3 = (\mathbf{k}_3, \omega_3) = \alpha_1 + \alpha_2, \quad \alpha_4 = (\mathbf{k}_4, \omega_4) = \alpha_1 - \alpha_2.$$

Further assume that the bispectra $B(\alpha_1, \alpha_2)$ and $B(\alpha_1, -\alpha_2)$ do not vanish; thus pressure components $p(\alpha_3)$ and $p(\alpha_4)$ coherent with $Z_1 Z_2$ and $Z_1 Z_2^*$ respectively are present. Thus the space- and time-dependent ensemble averages $\langle p(\alpha_4) Z_2 \rangle$ and $\langle p(\alpha_3) Z_2^* \rangle$ are both coherent with the long wave and further

$$\delta\tau_{\mathbf{k}_1} = i\mathbf{k}_2\{\langle p(\alpha_4) Z_2 \rangle - \langle p(\alpha_3) Z_2^* \rangle\} \tag{3.5}$$

appears to the long wave as a fluctuating tangential stress. Thus the momentum of the long wave will grow, according to

$$\frac{d}{dt}\mathbf{M}(\mathbf{k}_1) = \mathbf{k}_1 2\,\mathrm{Re}\left\langle \delta\boldsymbol{\tau} . \frac{\mathbf{k}_1}{|k_1|} Z_1 \right\rangle. \tag{3.6}$$

The reader can easily verify that (3.6) is identical to (2.33) to $O(\sigma_1/\sigma_2)$. (A further discussion is given in §4.2.) Closer agreement would not be meaningful, because the concept of averaging over a short time to obtain time-dependent means does not allow a finer resolution of σ_2 than $\sigma_2 \pm \sigma_1$.

Within this same interpretation, i.e. time-dependent averages over short times, we also see that the high-frequency mean fluxes associated with $\langle p_4 Z_4^* \rangle$ and $\langle p_3 Z_3^* \rangle$ would be indistinguishable from $\langle p_2 Z_2^* \rangle$. For short-time averages the sum of the three contributions would be accounted for as $\langle p_2 Z_2^* \rangle$ and therefore be interpreted as flux to the wave Z_2. However because of the broad bandwidth the uncertainty of the flux determined this way at high frequencies is just sufficient to account for the growth of the long wave due to the fluctuating stress. If we go to a finer resolution we shall find that the high-frequency waves actually grow more slowly than is indicated by the energy and momentum input at high frequencies, provided that the tangential stress causes growth at low frequencies.

Garrett & Smith (1976) have also studied the mechanism of Longuet-Higgins and came to the conclusion (their §5) 'that at most a fraction $(k_1 Z_1)$ of the total wind stress τ_s can go into long wave momentum'. Here τ_s is the rate of transfer of momentum to short waves. This statement could be misinterpreted by the casual reader, since τ_s is a short-time average and must not be confused with $\bar{\tau}_s$.

Consider the following case. Let the sum interaction supply Z_3 with energy and momentum at the rate

$$J^{(+)} = (\alpha_1 + \alpha_2) A \tag{3.7}$$

and let the difference interaction supply Z_4 at the rate

$$J^{(-)} = (\alpha_2 - \alpha_1) A^{(-)} = -(\alpha_2 - \alpha_1) A. \tag{3.8}$$

Thus let $A^{(-)} = -A^{(+)}$. Assuming there is no other input, the total flux at high frequency is

$$J = J^{(+)} + J^{(-)} = 2\alpha_1 A, \tag{3.9}$$

all of which appears as long-wave energy and momentum. Garrett and Smith's statement is correct, but τ_s is the modulus of (3.5), thus τ_s is the modulus of the fluctuations around $\bar{\tau}_s$, which is indeed for our special case exactly a factor $(k_1 Z_1)^{-1}$ larger than the momentum component of (3.9). Garrett & Smith go on to say that in the relation $\dot{\mathbf{M}} \leqslant (k_1 Z_1)\, \tau_s$ equality is achieved only if $\tau_s(x,t)$ is a series of δ-functions at the long-wave crests. This statement is correct only if $\tau_s(x,t) \geqslant 0$ holds, but this side condition is not obviously satisfied. Even if we postulate that a reversal of the short-time-averaged tangential stress is impossible, such a postulate can be meaningful only if scaling conditions are satisfied. It must at least be possible to define a stable small-scale-averaged stress tangential to a (small-scale-averaged) mean surface, and this requires a spectral gap between high- and low-frequency waves. For a typical wave spectrum there is no gap and while the concept of a fluctuating horizontal (not tangential) stress can still be used a phase average cannot be defined; the equivalent information must now be obtained from cross-spectra, bispectra etc.

If, then, $\tau_s(x,t)$ is the fluctuating time series $\langle p(x,t)\, \zeta_{,x}(x,t)\rangle_{TL}$ averaged over short times T and lengths L centred on (x,t), then the generalization of Garrett & Smith's inequality is the very weak statement

$$\dot{\mathbf{M}}(\mathbf{k}) \leqslant k \langle \zeta\zeta\rangle^{\frac{1}{2}}(\mathbf{k}) \langle \tau_s \tau_s\rangle^{\frac{1}{2}}(\mathbf{k}). \tag{3.10}$$

Note that $\langle \tau_s, \tau_s\rangle$ can be defined as stable variance only around a stable mean $\overline{\tau_s(x,t)} = \tau$, so that $\langle \tau_s, \tau_s\rangle$ contains all the large fluctuations due, for instance, to the real part of the low-frequency pressure field. We also see that the bispectral formulation is the natural generalization of Longuet-Higgins' mechanism to a statistical field, and that the order of magnitude is not easily estimated, (3.10) being virtually useless, and in any case being a much weaker statement than (2.32) and (2.33).

The method of Garrett & Smith (1976) of slicing the low-frequency spectrum into bands, treating each band by itself and letting it interact with high frequencies is essentially an approximated bispectral analysis. The interaction of waves at similar wavenumbers is lost in this treatment, but this presumably introduces no large error as discussed in §4.2.

3.3. *Orders of magnitude*

Certainly for truly infinitesimal waves the bispectral transfer expressions should be negligible compared with the usually considered cross-spectral mechanism. Thus if δ is an average wave slope we should estimate

$$S_2 \approx \delta^2 S_1, \tag{3.11}$$

where S_2 is the bispectral and S_1 the cross-spectral source function. This estimate follows from the concept that S_2 is proportional to $\langle p^{*(2)} \dot{\zeta}^{(2)}\rangle$, where the superscript (2) indicates a second-order approximation in the wave slope. Thus

$$p^{(2)} \approx \delta p^{(1)}, \quad \zeta^{(2)} \approx \delta \zeta^{(1)} \tag{3.12}$$

and (3.11) follows. This estimate is crude; it may be quite unreliable even for infinitesimal waves if the structure of the wave spectrum is taken into account. The reason for this is that the relevant coupling parameter is not δ, but rather $\tilde{\delta} \simeq (k_2 Z_1)$, with Z_1 a low-frequency wave amplitude and k_2 a wavenumber typical of the high-frequency end of the spectrum. Of course $\tilde{\delta}$ must be averaged somehow over the spectrum. A typical value for $\tilde{\delta}$ is $\tilde{\delta} \simeq 1$. This is the main reason why our model calculations in §4 yield

$$S_2 \lesssim \tilde{\delta} S_1. \tag{3.13}$$

The reader will of course suspect that the theory collapses for $\tilde{\delta} \simeq 1$; this is not quite so. We are still considering the first term of a convergent expansion, but the approximation is poor. The situation closely resembles approximating $\sin \tilde{\delta}$ by $\tilde{\delta}$ for values of $\tilde{\delta} \approx 1$.

4. A model calculation

4.1. *Bispectral models*

Practically nothing is known about the bispectra. We shall now explore some hypotheses in order to arrive at order-of-magnitude estimates. For simplicity and for the lack of an alternative, we shall reluctantly rely on models based on the assumption that the pressure can be expanded in the wave slope. This approach is used as a guideline only, and should not be mistaken as an attempt to formulate a consistent theory.

Let $U(z)$ be the mean wind speed, let $u_i(\sigma_1)$, $u_i(\sigma_2)$, $p(\sigma_1)$ and $p(\sigma_2)$ be wave-induced variables linear in $\zeta(\sigma_1)$ and $\zeta(\sigma_2)$, and let $u_i(\omega_3)$ and $p(\omega_3)$ be coherent with $\zeta(\sigma_1)\,\zeta(\sigma_2)$. The problem can then be cast into the following framework (see, for example, Hasselmann 1968).

The linear response is described by

$$u_3(\sigma_1, z = 0) = i(U - c_1)\,\kappa_1 \zeta, \tag{4.1}$$

$$Lu_3(z) = 0, \quad u_3(z \to \infty) = 0. \qquad (4.2), (4.3)$$

Here L is assumed to be a modified Orr–Sommerfeld operator, still of second order but in some way or other including effects of turbulent stresses. The solution of

$$Lu_3 = 0, \quad u_3(0) = 1, \quad u_3(\infty) = 0 \tag{4.4}$$

is then assumed to yield satisfactory results for the linear problem; thus with

$$\mathbf{k}_i = (k_i, 0) \quad \text{and} \quad |k_i| = \kappa_i,$$

$$p(z = 0) = iT_p(k, \omega), \quad u_1(z = 0) = iT_1(k, \omega) \tag{4.5}$$

with
$$T_p = \begin{cases} (U - \omega/\kappa) - i\mu\omega/\kappa & \text{for} \quad \omega/\kappa > U, \\ (U - \omega/\kappa) & \text{for} \quad \omega/\kappa < U. \end{cases} \tag{4.6}$$

The transfer function is chosen in such a way that the resulting linear growth rates follow the empirical relation given by Snyder & Cox (1966):

$$\frac{dF}{dt} = \frac{\rho_a}{\rho}\mu\left(\frac{U}{c} - 1\right)\sigma F. \tag{4.7}$$

Although the value of μ varies from experiment to experiment between $\mu = 0{\cdot}1$ and $\mu = 1{\cdot}0$, this value is of no consequence for this discussion since we shall be interested only in the ratio of nonlinear to linear effects. Thus we set $\mu = 1{\cdot}0$.

The transfer function T_1 is not uniquely determined by the growth rate, but we shall assume that

$$ik_1(U-c)\,u_1(0)+u_3\,dU(z=0)/dz = -ik_1 p(z=0) \tag{4.8}$$

is still approximately correct for our modified linear model, thus assuming that turbulent stresses play only a minor role in (4.8). We can then distinguish three nonlinear effects.

(*a*) A nonlinear surface wave will excite a response in the same way as a linear surface wave.

(*b*) Nonlinear boundary conditions in the absence of a nonlinear surface wave will have a similar effect to a nonlinear surface wave.

(*c*) Nonlinear stresses q_{ij} will give rise to nonlinear pressure amplitudes.

Mathematically it is unreasonable to distinguish between (*a*) and (*b*). However, we keep them apart because the effect of (*a*) is already completely described by the linear theory of the atmospheric response, the pressure transfer coefficient being assumed to be known.

All calculations will be performed with a simplified spectrum for the linear wave field, assumed to be unidirectional:

$$F(\sigma) = \begin{cases} \tfrac{1}{2}\alpha g^2\sigma^{-5} & \text{for} \quad \sigma_p \leqslant \sigma \leqslant \sigma_c, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

For Phillips' constant α we shall use $\alpha = 10^{-2}$. For a unidirectional spectrum in which all waves propagate in the x direction, i.e.

$$\mathbf{k}_i = (k_i, 0), \quad \omega_i = \sigma(k_i) > 0 \quad \text{for} \quad k_i > 0, \tag{4.10}$$

the product of the coupling constant D and the propagator becomes very simple:

$$\frac{\sigma_3^2}{\sigma_3^2-(\omega_1+\omega_2)^2} D(\alpha_3, \alpha_1, \alpha_2) = \frac{\sigma_3^2}{g}\nu_1\nu_2. \tag{4.11}$$

Here $\alpha_i = (k_i, \omega_i)$, $\alpha_3 = -(\alpha_1+\alpha_2)$, $\sigma_3 = \sigma(k_3)$ and $\nu_i = \omega_i/\sigma_i = k_i/|k_i| = \pm 1$.

4.2. *A brief discussion of three models*

We have done calculations for three models A, B and C.

Model A. We assume that (*a*) is the only relevant nonlinear effect. The wind field notices the presence of nonlinear forced waves, but nonlinear wind dynamics produced by (*b*) and (*c*) are neglected.

Model B. The wind is driven by nonlinear stresses obtained from the linear solution and the effects of nonlinear boundary conditions and nonlinear waves are neglected. This model produced only small fluxes and growth rates and is not discussed in any detail. We mention only that it can damp waves outrunning the wind, even though the underlying linear model leaves the waves unaffected in this case.

Model C. The nonlinear boundary conditions dominate the atmospheric response, so that the higher wind velocity on the crests of long waves produces an input to short waves which is higher on the crests than in the troughs.

Some properties are common to all three models:

(i) The fluxes contribute over a wider frequency band than that over which the growth occurs. Thus wave growth and decay can occur only in the range $\sigma_p \leqslant \sigma \leqslant \sigma_c$ according to (4.9), whereas the sum interaction results in flux in the range $2\sigma_p \leqslant \sigma \leqslant 2\sigma_c$ and the difference interaction yields flux in the range $0 \leqslant \sigma \leqslant \sigma_c - \sigma_p$.

(ii) The interaction integrals have considerable contributions from the interaction of long waves and short waves.

The second property means that the models yield a large contribution in the range where they are most suspect. (The situation is somewhat complex because the wave spectrum falls off so rapidly.) Thus it turns out that the model estimates of Garrett & Smith (1976) are probably quite reasonable. Their treatment cannot properly deal with the interaction of two waves with similar wavelengths – in this range their analysis is inferior to ours – but contributions from this range are not dominant.

For waves with widely separated wavenumbers their analysis is clearly superior to ours, since wave modulation is accounted for with conservation of action: a much better approximation than ours, which includes only second-order interactions. In one respect our treatment is better, even in this case, because we can consider the detailed wavenumber–frequency structure of the atmospheric response instead of only small-scale-averaged properties. A treatment which combines the virtues of both approaches should be possible and is planned for the future. In light of these shortcomings it is quite surprising, and possibly only a fortunate coincidence, that our models yield the same order of magnitude for the low-frequency growth rates as Garrett & Smith's estimates, namely roughly 30 % of the linear input; this is also discussed in the summary.

In spite of the questionable quality of our models we show two results to give some impression of how the bispectral mechanism might work. We give only a brief discussion; and the reader is referred to a longer earlier version of this paper for details of the calculations (Hasselmann 1976).

Considering two wavenumbers k_1 and k_2 with $k_1 \ll k_2$, we first note that interaction produces two forced waves in the side bands $\alpha_3 = (k_1+k_2, \sigma_1+\sigma_2)$ and $\alpha_4 = (k_2-k_1, \sigma_2-\sigma_1)$. From (4.11) we have

$$Z(\alpha_3) = -Z(\alpha_4)\{1+O(k_1/k_2)\} \tag{4.12}$$

and

$$Z(\alpha_3) = k_2 Z(\alpha_1) Z(\alpha_2)\{1+O(k_1/k_2)\}. \tag{4.13}$$

For $k_2 Z_1 > 1$, (4.13) is only a poor approximation to the WKBJ solution, and $Z(\alpha_3)$ and $Z(\alpha_4)$ are considerably overpredicted in magnitude, which results in the above-mentioned deterioration of our models in the range $k_2 \gg k_1$. Nevertheless the relative phases are correctly given by (4.12) and this allows us to predict at least some qualitative features of models A and C.

For model A the pressure amplitude $p = p_1 + ip_2$ has imaginary components

$$p_2(\alpha_3) = \mu\omega_3(U-c_3)\, Z(\alpha_3) \tag{4.14}$$

and

$$p_2(\alpha_4) = \mu\omega_4(U-c_4)\, Z(\alpha_4) \tag{4.15}$$

with $\mu = 1$ for $U-c \geqslant 0$ and $\mu = 0$ otherwise. For $\sigma_2 \gg \sigma_1$ the pressure transfer coefficient is nearly constant for all three bands α_2, α_3 and α_4. What happens in these circumstances can be described in two ways:

(i) The sum and difference fluxes both feed into the wave system at nearly equal rates. Thus the high-frequency wave $Z(\alpha_2)$ receives practically all of the input, while at low frequencies the growth due to the sum interaction is cancelled by the decay due to the difference interaction. We have the opposite case to the one considered in (3.7)–(3.9). (Note that the product $D \times$ propagator is squared in (2.36) for model A.) We call model A an odd model because the bispectrum $B_2(\alpha_2, \alpha_1)$ is an odd function of α_1.

(ii) In the fluctuating-stress treatment we do not see easily how things balance at high frequencies, but in any case the fluctuating stress is obtained from (3.5) as

$$\delta\tau(k_1) = 2(k_2 Z_1)\,\pi_2 \sin(k_1 x - \sigma_1 t) \tag{4.16}$$

with π_2 related to the (in our model unspecified) real part of the high-frequency pressure by $\pi_2 = 2k_2\, p_1(\alpha_2)\, Z_2$. The stress fluctuates out of phase with $Z(\alpha_1)$ and thus does not contribute to long-wave growth.

Touching again on the arguments presented after (3.9), we note that, even in a situation in which we could define a phase average, there would be no obvious relation between the magnitude of the fluctuations $\delta\tau$, which are of order $k_2 Z_1 \pi_2$, and the mean input, which is determined by the input integral over all frequencies. Even for Z_1 sufficiently small, so that $k_2 Z_1 < 1$, the postulate that the small-scale-averaged stress should not reverse would imply stringent nonlinear side conditions for the atmospheric response. However, the first impression is that there is no dynamical basis for such a postulate. One can consider model situations, for instance a spectrum with two very narrow peaks, one at high and one at low frequencies. By varying the energy in the peaks, their separation and $\tau = \rho^{(a)} u^{*2}$, it should be fairly simple to decide whether the postulate is physical or not, since the real part of the pressure amplitude can be predicted with reasonable accuracy, perhaps even the imaginary part (Miles 1957).

We now turn to model C, which in a sense turns out to be the opposite of A. Considering only the nonlinear boundary conditions and keeping all terms to second order (which for $k_2 \gg k_1$ and $Z_1 \gg Z_2$ is a very poor approximation), we obtain

$$u_3(\omega_3) = -ik_3\{u_1(\sigma_1)\, Z(\sigma_2) + u_1(\sigma_2)\, Z(\sigma_1)\}_{z=0} \tag{4.17}$$

and a similar expression for $u_3(\omega_4)$.

In (4.17) we relate $u_1(\sigma)$ to $Z(\sigma)$ by

$$u_1(\sigma, z=0) = (U-c)\,|k|\,Z(\sigma), \tag{4.18}$$

so that $u_3(\omega_3)$ is related to $Z(\sigma_1)\,Z(\sigma_2)$, and finally use $p = iT_p\, u_3(z=0)$ as given by (4.6). To $O(\sigma_1/\sigma_2)$ we find

$$p_2^C(\omega_3) = p_2^A(\omega_3), \quad p_2^C(\omega_4) = -p_2^A(\omega_3). \tag{4.19}$$

(These results obtained for $\sigma_1 \ll \sigma_2$ were then used for all (σ_1, σ_2) so that the results of model A could be used with a simple change of sign in the difference interaction terms.)

In (4.19) the equality in size is surprising, but the change from an odd to an even model is not. Whereas the bispectrum of model A must by its construction notice on which side of the resonance curve the point $(\mathbf{k}, \omega)$ lies, the bispectrum of model C is insensitive to the resonance curve.

Thus model C behaves like the special case (3.7)–(3.9) and the discussion given there applies: the fluctuations $\delta\tau$ are now in phase with the long wave, and $|\delta\tau|$ is of order $(k_2 Z_1)\,\tau_2$, where τ_2 is the cross-spectral linear momentum flux to Z_2,

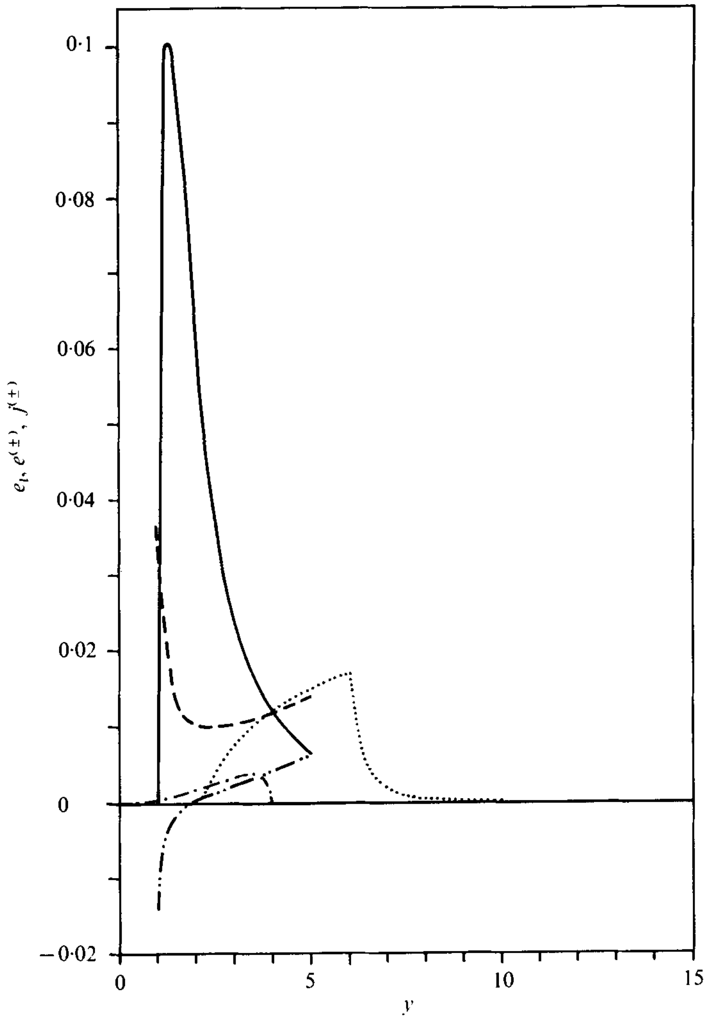

FIGURE 2. Normalized energy growth rates and fluxes for model A, calculated for wave spectrum (4.9) with $\alpha = 0{\cdot}01$, $b = \sigma_c/\sigma_p = 5$ and $y_0 = c_{\text{peak}}/U = 1{\cdot}0$. ——, e_l, Snyder–Cox input; - - - -, $e^{(+)}$; — · · · —, $e^{(-)}$; · · · · ·, $j^{(+)}$; — · —, $j^{(-)}$.

i.e. $\tau_2 = 2k_2 Z_2 p_2$. Thus, although the underlying approximations and manipulations of model C can hardly be defended, the size of the fluctuations is admittedly large, but not completely unrealistic.

In figures 2 and 3 we show the non-dimensional energy growth rates e and $e^{(\pm)}$ and energy fluxes j and $j^{(\pm)}$, where

$$e = e^{(+)} + e^{(-)}, \quad j = j^{(+)} + j^{(-)}, \tag{4.20}$$

$(\pm)$ indicating sum and difference interaction contributions, and

$$E(\sigma)\, d\sigma = \gamma e(y)\, dy, \quad J_E(\sigma)\, d\sigma = \gamma j(y)\, dy, \tag{4.21}$$

with $y = \sigma/\sigma_p$, $\gamma = \alpha U^3$ and α and σ_p as defined in (4.9). The growth rates and fluxes e and j also depend on the high-frequency cut-off σ_c through the parameter $b = \sigma_c/\sigma_p$

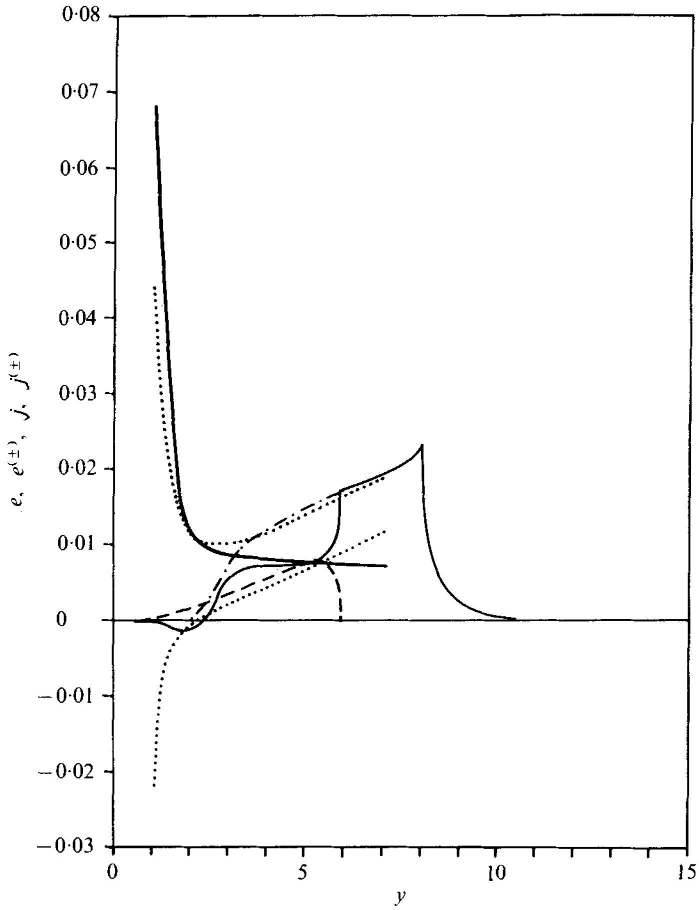

FIGURE 3. Models A and C for $b = 7$ and $y_0 = 1{\cdot}0$ with the other parameters as before. The linear input (not shown) can be taken from figure 2. ——, $e(C) = e^{(+)}(A) - e^{(-)}(A)$; ——, $j(C) = j^{(+)}(A) - j^{(-)}(A)$; ·····, $e^{(+)}(A)$, $e^{(-)}(A)$, upper curve is $e^{(+)}(A)$; –·–, $j^{(+)}(A)$; -----, $j^{(-)}(A)$. $j^{(+)}(A) = j(C)$ for $y > b-1$. Model C, corresponding closely to Longuet-Higgins' fluctuating-stress mechanism, shows large bispectral input at high, forced frequencies, balanced by growth concentrated at low, free frequencies amounting to 70 % of the linear input.

and on the ratio of the wind velocity to the phase speed at the spectral peak through $y_0 = \omega_0/\sigma_p = c_p/U$.

In figure 2 the growth rate $e_l(y)$ according to the underlying linear Snyder–Cox model (1.2) or (4.6) is also shown. All values are proportional to μ and we have set $\mu = 1$. Figure 2 shows the expected behaviour of the growth rates for model A, although some growth $e = e^{(+)} + e^{(-)} \approx 0{\cdot}2e_l$ does remain at low frequencies as a result of the interaction of waves not widely separated in wavenumber.

Figure 3 shows the results for both models, now for a value of $b = 7$. Model C in particular demonstrates how strongly the distribution of growth rates and fluxes may differ.

At high forced frequencies we observe a large input and of course here there can be no growth; this is the input range. At intermediate frequencies we find a balanced range where the input approximately balances the growth. At low frequencies, there is hardly any, or even negative, input; this is the growth range. The input at high frequencies approximately balances the growth at low frequencies.

The cusps of j reflect the low- and high-frequency cut-offs in the spectrum. For $y \geqslant b-1$ the difference flux vanishes and for $y \geqslant b+1$ the low-frequency peak can no longer contribute to the flux. This (together with phase-space effects) results in the rapid decay of $j^{(+)}$ for $y > b+1$. (We have not shown model C for $b = 5$, but the reader can easily obtain the results for this case by changing the sign of $j^{(-)}$ and $e^{(-)}$, thus producing $e(C) = e^{(+)}(A) - e^{(-)}(A)$ and $j(C) = j^{(+)}(A) - j^{(-)}(A)$ in figure 2.) We may have overemphasized the role of interactions between short waves and long waves. This has already been mentioned above for the low-frequency growth of model A. Another indication is that, in figure 3, $j^{(+)}$ at $y = b+2 = 9$ and $j^{(-)}$ at $y = b-2 = 5$ are not approximately equal as would be the case if long–short wave interactions dominated the rest.

It is evident that for both models the bispectral input is typically 30 % of the linear input and thus not negligible. An odd model such as model A would be of but little help should the JONSWAP balance encounter difficulties, since it provides input and growth mainly at high frequencies. In contrast, an even or Longuet-Higgins-type model such as model C would fit in very nicely as it would provide growth at low frequencies in addition to the measured input fluxes at these frequencies.

The models are certainly very crude and only qualitative, but they do indicate that the bispectral mechanism and specifically its even components corresponding to Longuet-Higgins' mechanism cannot be discarded as inconsistent with gross order-of-magnitude estimates based on $u^*/U \approx \frac{1}{30}$. This result is in agreement with that of Garrett & Smith (1976), but whereas their result does not imply any correspondence between a linear pressure input and a fluctuating stress input, our analysis indicates that an increase in the linear mechanism should also result in larger stress fluctuations and thus also yield an increase in the bispectral input. Concerning the relative merits of our analysis and Garrett & Smith's (1976), an effort to combine the good features of both seems to be worthwhile.

5. Comparison with experiment

A direct comparison with experiments is not possible, because bispectra have not yet been analysed to the author's knowledge. The models were presented only for order-of-magnitude estimates and not for prediction of flux distributions. The discrepancies between existing measurements certainly indicate that the atmospheric response is highly nonlinear. Some, but not all, of the discrepancies between Dobson's (1971), Elliott's (1972) and Snyder's (1974) measurements could be explained by bispectral contributions, but the role of bispectral contributions remains speculative until bispectra have been obtained.

6. Summary

It has been demonstrated that the pressure field can transfer energy and momentum to free waves by acting on non-resonant forced waves. The transfer is proportional to the bispectrum $\langle p\zeta\zeta\rangle$. The magnitude of the effect has been estimated by a calculation for a simplified model of the bispectra, and found to be non-negligible (typically 30 % of the linear input) in this case.

The distribution of energy and momentum input fluxes may differ strongly from the distribution of the resulting growth rates, the input lying at higher frequencies than the growth. The effectiveness of bispectral contributions remains uncertain until bispectra have been analysed. Should it be established that a linear input cannot provide the necessary JONSWAP minimum energy and momentum flux in the low-frequency range of the wave spectrum, the bispectral contributions are obvious candidates for closing the gap.

It should be mentioned that Garrett & Smith (1976) arrived at an input to long waves of the order of 4 % of the total stress τ due to the fluctuating-stress mechanism. This model would roughly correspond to our model C, and for a linear input to the waves of the order of $0{\cdot}1$–$0{\cdot}2\tau$, this is the same order of magnitude as was obtained in our model calculation. By a quite different line of argument Valenzuela & Wright (1976) also arrived at the same order of magnitude for Longuet-Higgins' fluctuating-stress mechanism.

We agree with Garrett & Smith (1976) that measurements of fluctuating stress over waves are desirable, but add that measurements of waves and pressure to high ($f \sim 2$ Hz) frequencies with series long enough to analyse bispectra will be necessary for a consistent analysis in terms of a fluctuating stress.

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Appendix. Feynman diagrams and the symmetry of the coupling coefficients

Feynman diagrams

We are looking for a formal solution of the integral equation

$$(-\omega^2+\sigma^2)\,Z(-\alpha_1) = \sigma^2\{\epsilon b(-\alpha_1)+d(-\alpha_1) + \sum_{n=3}^{\infty}\int D(\alpha_1,\ldots,\alpha_n)\,Z(\alpha_2)\ldots Z(\alpha_n)\,\delta(\alpha)\,d\alpha_2,\ldots,d\alpha_n\} \quad \text{(A 1)}$$

with

$$\alpha_i = (\mathbf{k}_i,\omega_i), \quad \alpha = \sum_{i=1}^{n}\alpha_i.$$

(For $b \not\equiv 0$ and $d \equiv 0$, (A 1) is the general nonlinear dispersion relation in the sense of generalized Kramers–Kronig relations obtained from the causality requirement that $Z(\mathbf{k},\omega)$ should have no singularities for $\operatorname{Im}\omega > 0$.) Here $d(\alpha)$ is determined by the initial conditions and $\epsilon b(\alpha)$ is a prescribed weak external forcing field. The solution can then be obtained in powers of the fields $\epsilon b+d$ by iteration, i.e.

$$\left.\begin{aligned} Z^{(0)} &= -\frac{\sigma^2}{\omega^2-\sigma^2}(\epsilon b+d),\\ Z^{(n)} &= Z^{(0)}+D\{Z^{(n-1)}\}\left(\frac{-\sigma^2}{\omega^2-\sigma^2}\right),\end{aligned}\right\} \quad \text{(A 2)}$$

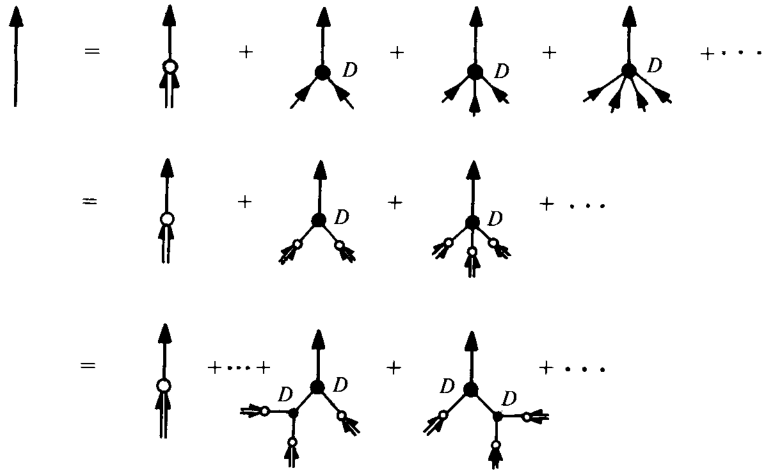

FIGURE 4. The integral equation for $Z(k, \omega)$ and the iterative procedure. Products of terms occurring in the series give rise to Feynman diagrams like, for instance, figure 1(*b*).

and this process can be visualized with diagrams; see figure 4. Here the single arrow stands for the solution Z while the double arrow indicates the field $d+\epsilon b$. Figure 4 can be read both as the integral equation (A 1) and as the iteration procedure (A 2). The rules for these diagrams are obvious and need not be given here, since this presentation is exactly equivalent to the one given by Hasselmann (1966*a*, *b*).

The symmetry of the coupling coefficients

The fact that the equations of motion for surface gravity waves can be cast into a Hamiltonian form (Hasselmann 1966*a*; Whitham 1966) has the immediate consequence that the equations of motion take the form

$$\dot{a}^{\nu}_{\mathbf{k}} + i\nu\sigma(k)\, a^{\nu}_{\mathbf{k}} = \Sigma H^{-\nu\nu_1\ldots\nu_n}_{-\mathbf{k}\mathbf{k}_1\ldots\mathbf{k}_n}\, a^{\nu_1}_{\mathbf{k}_1}\ldots a^{\nu_n}_{\mathbf{k}_n}, \tag{A 3}$$

with the coupling coefficients H symmetrical to permutations of the indices $(-\mathbf{k}, -\nu; \ldots; \mathbf{k}_n, \nu_n)$. This seems to close this subsection. However, it should be noted that the $a^{\nu}_{\mathbf{k}}$ as used in (A 3) are not easily observable variables. To lowest order we have

$$a^{\nu}_{k} = A^{\nu}_{k} = \dot{\zeta}k/\sigma - i\nu\zeta k \tag{A 4}$$

but if we wish (A 3) to be valid up to any order then

$$A^{\nu}_{k} = a^{\nu}_{k} + \sum_{n=2}^{\infty} C^{-\nu\nu_1\ldots\nu_n}_{-kk_1\ldots k_n}\, a^{\nu_1}_{k_1}\ldots a^{\nu_n}_{k_n}, \tag{A 5}$$

or

$$a^{\nu}_{k} = A^{\nu}_{k} + \sum_{n=2}^{\infty} \hat{C}^{-\nu\nu_1\ldots\nu_n}_{-kk_1\ldots k_n}\, A^{\nu_1}_{k_1}\ldots A^{\nu_n}_{k_n}. \tag{A 6}$$

If we wish (A 3) to be valid up to some order m then also in (A 5) we need only terms up to order m. It is simple to find the defining equation for the C's; this follows from the Hamiltonian formulation, but will not be discussed here in any detail; see also the last subsection of this appendix. We wish only to emphasize that the price paid for (A 3) is a complication in terms of the observable variables $A^{\nu}_{\mathbf{k}}(t)$. If, for instance, the

$A_{\mathbf{k}}^{\nu}$ are Gaussian variables, the $a_{\mathbf{k}}^{\nu}$ are not. Thus statistical averages $\langle a_{\mathbf{k}_1}^{\nu_1} a_{\mathbf{k}_2}^{\nu_2} a_{\mathbf{k}_3}^{\nu_3} a_{\mathbf{k}_4}^{\nu_4} \rangle$ will not vanish if $\mathbf{k}_1, \ldots, \mathbf{k}_4$ cannot be split into complex-conjugate pairs, but will contain sixth-order contributions $\langle A_{\mathbf{k}_1}^{\nu_1} \ldots A_{\mathbf{k}_6}^{\nu_6} \rangle$. This complicates the use of diagram techniques and symmetry properties of the C's have to be studied.

Of course, this does not answer the question for which wave field the $a_{\mathbf{k}}^{\nu}(t)$ or the $A_{\mathbf{k}}^{\nu}(t)$ are better approximated as Gaussian. (According to Newell (1968) the transfer expressions are valid whether or not the field is Gaussian. Newell did not consider off-shell dependence of the coupling, and it is not clear to the author whether this influences his results.)

In any case it is useful to know the symmetries of the coupling whenever we wish to express the transfer expressions in terms of directly observable variables, or especially, whenever we have external coupling. We proceed to show that the equation of motion (2.2), or equivalently (A 1), has the property that the $D(\alpha_1, \ldots, \alpha_n)$ are symmetric functions of $(\alpha_2, \ldots, \alpha_n)$. The $D(\alpha_1, \ldots, \alpha_n)$ are identical with the frequency-independent coupling coefficients if all frequencies lie on the dispersion shell, but differ from these otherwise.

Proof of the symmetry

We do not give an extensive constructional proof of the symmetry of $D(\alpha_1, \ldots, \alpha_n)$, and in fact only prove the following lemma.

Lemma. If assumption (A) below is valid then the symmetry of $D(\alpha_1, \ldots, \alpha_n)$ follows.

(A) We assume that energy and momentum of the wave field both remain steady if the spectrum remains steady at all wavenumbers.

Remark. The intention of introducing (A) is to ward off all arguments about how the energy and momentum of a free wave should be defined. In fact, (A) is not nearly so innocent as it may appear at first sight. It is, for example, conceivable that the wave energy grows owing to growing bispectra, with spectra remaining stationary. However, we shall not dwell on this aspect here.

We now give an outline of the proof, before the very simple idea gets lost in detailed considerations. The surface elevation $\zeta(x, t)$ is a well-defined variable, the location of the air/water density discontinuity. Just as well defined, though more difficult to measure, is the atmospheric pressure at the surface $p^{(a)}$. Although the energy and momentum of a wave field are fairly complicated functions of the wave height the fluxes of energy and momentum across the interface are simple. The instantaneous fluxes of energy and momentum averaged over a horizontal area L^2 are given by

$$J_E(t) = -\langle p(t)\, \dot{\zeta}(t) \rangle_L, \tag{A 7}$$

$$J_i(t) = \langle p(t)\, \zeta_{,i}(t) \rangle_L. \tag{A 8}$$

(Perhaps one might have expected the length of the surface normal $[1 + (\zeta_{,i}\, \zeta_{,i})]^{\frac{1}{2}}$ to appear in (A 7) and (A 8), but this factor cancels exactly with the projection of the surface length ds onto the horizontal dx; $dx/ds = [1 + (\zeta_{,i}\, \zeta_{,i})]^{-\frac{1}{2}}$.)

Now the idea of the proof is simple. We prescribe an external forcing, so that the waves do not grow (up to an also prescribed order). This is achieved by suitable non-resonant forcing. After an initial transient build-up time for the forced waves, the wave field becomes steady. (At least the spectrum becomes steady.) Thus the energy and momentum become steady according to assumption (A). Therefore the time averages $\langle J_E(t) \rangle_T$ and $\langle J_i(t) \rangle_T$ must vanish. This does not mean that the spectral contributions to these fluxes vanish individually at all wavenumbers concerned, only that the total sum vanishes. This can then be shown to be equivalent to the symmetry of the coupling.

As the initial state we now take $\dot{\zeta} \equiv \zeta \equiv 0$; thus in (A 1), $d(\alpha) \equiv 0$. The idea is now to force the field with n pressure components b_i so that the pressure field has n poles at the n locations $\alpha_i = (\mathbf{k}_i, \Omega_i)$:

$$b(\mathbf{k}_i) = q_i/(\omega_i - \Omega_i + i\epsilon). \tag{A 9}$$

The pole strength q_i can be chosen arbitrarily, but the locations α_i must satisfy the constraint

$$\sum_{i=1}^{n} \alpha_i = 0. \tag{A 10}$$

For simplicity we shall now introduce additional constraints on the α_i, which should and can be removed step by step for a complete proof. (The reader is however informed at this point that this constraint removal will not be carried through here, but we shall see how it can be done in principle.) These additional constraints are that

(i) no partial sum of the α_i shall vanish, i.e.

$$\sum_{j \in L} \alpha_j \neq 0 \quad \text{unless} \quad L = \{1, \ldots, n\}, \tag{A 11}$$

(ii) no wave shall be excited in resonance, i.e.

$$\Sigma\Omega_i \neq \sigma(\Sigma\mathbf{k}_i) \quad \text{for all partial sums.} \tag{A 12}$$

Now the solution Z can be expanded as a power series in the n variables $q_1, \ldots, q_n$. Because of (A 12) there will be no growth of the wave spectrum proportional to the power $q_1 \ldots q_n$. Thus the total energy and momentum input proportional to $q_1 \ldots q_n$ must also vanish. (This argument involves rather strong assumptions concerning the expansion of the solution in a power series in the q_i. There may of course be unsteadiness due to wave–wave interactions of the excited free waves, if for example

$$k_1 + k_2 + k_3 + k_4 = 0, \quad \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 \neq 0 \quad \text{but} \quad \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0.$$

None of this unsteadiness can, however, be proportional to $q_1 \ldots q_n$. For the anxious reader we could also rule out these wave–wave interactions by carefully selecting the $\mathbf{k}_i$, but this would severely shrink the phase space available to us.)

The input consists of n terms. If we denote by $Z_1^{(n-1)}$ the wave component proportional to $q_2 \ldots q_n$, and likewise by $Z_i^{(n-1)}$ the one proportional to $(q_1 \ldots q_n)/q_i$, then the total energy and momentum transfer is given by

$$\lambda \Sigma \alpha_i q_i Z_i^{(n-1)} = 0 \tag{A 13}$$

(λ is a constant factor, which we may have overlooked). Defining the total coupling of $(q_1 \ldots q_n)'_i$ (here $(\;)'_i$ indicates that q_i is missing in the product) in Z_i by $\tilde{G}^{(n-1)}(\alpha_i; (\alpha_1 \ldots \alpha_n)'_i)$, or more briefly $\tilde{G}^{(n-1)}(\alpha_i, (\alpha)'_i)$, so that

$$Z_i^{(n-1)} = \tilde{G}^{(n-1)}(\alpha_i, (\alpha)'_i)\,(q_1 \ldots q_n)'_i, \tag{A 14}$$

we have from (A 13)

$$\sum_{i=1}^{n} \alpha_i \tilde{G}^{(n-1)}(\alpha_i; (\alpha)'_i) \equiv 0 \tag{A 15}$$

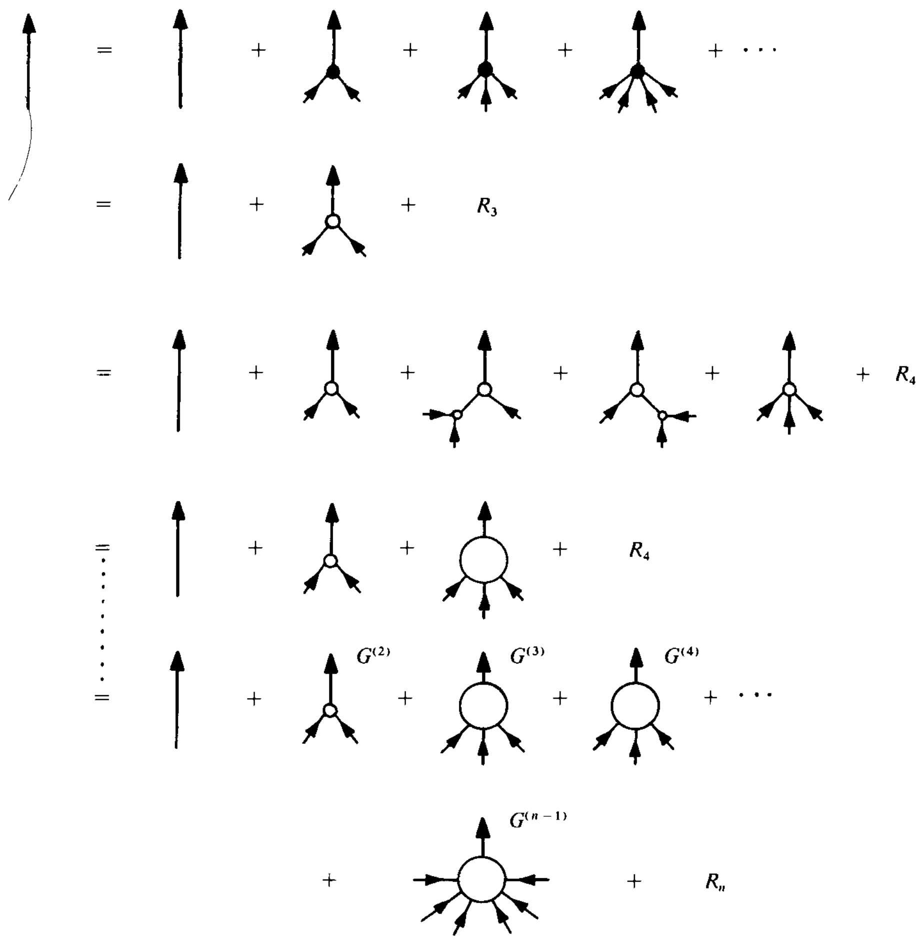

FIGURE 5. Diagram showing the definition of the vertex functions $G^{(n)}$. (The number of legs on the blobs should not be counted for $n > 3$.)

for all α_i satisfying the constraints (A 10)–(A 12). Now the proof involves two more steps.

Step 1. Here we show

(i) that $\tilde{G}^{(n-1)}$ may be written as

$$\tilde{G}^{(n-1)} = \Pi^{(n)}\{\Gamma^{(n-1)} + D(\alpha_1, \ldots, \alpha_n)\} = \Pi^{(n)}G^{(n-1)}, \tag{A 16}$$

with $\Pi^{(n)}$ a symmetric function of $(\alpha_1, \ldots, \alpha_n)$,

(ii) that if $\tilde{G}^{(n-1)}$ is symmetric for all $n' < n$, then $\Gamma^{(n-1)}$ is also symmetric.

Step 2. If step 1 is taken, then D also satisfies

$$\Sigma\alpha_i D^{(n)}(\alpha_i, (\alpha)'_i) \equiv 0, \tag{A 17}$$

and we shall show that this implies the symmetry of D.

Thus, if both step 1 and step 2 are taken we have proved by induction that both $G^{(n-1)}$ and $D^{(n)}$ are symmetric functions of their arguments, since we can anchor the proof at $n = 3$, for instance with the aid of (2.12).

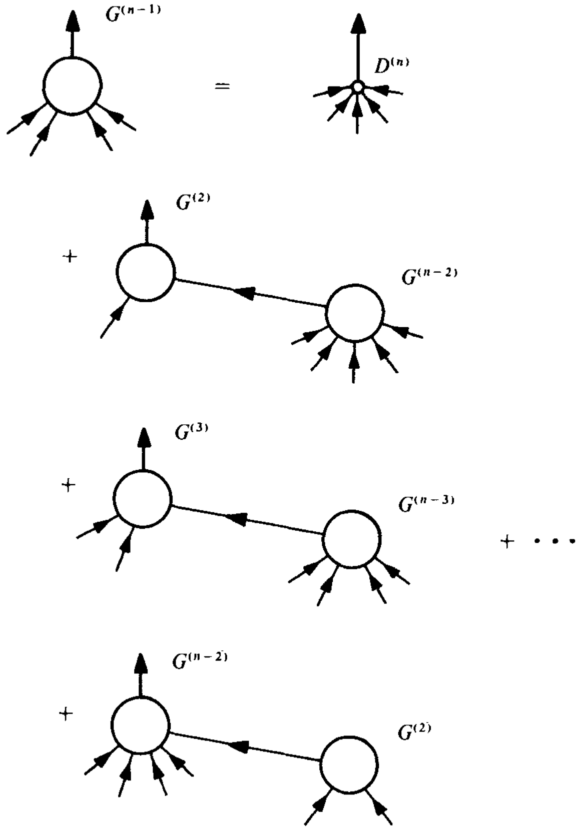

FIGURE 6. The structure of the vertex functions. (Number of legs on blobs schematic only.)

Step 1. Figures 5 and 6 show the structure of the relevant diagrams. First, a forcing factor q_i gives rise to

$$Z^{(1)} = \tilde{G}^{(1)}(-\alpha_i, \alpha_i)\, q(\alpha_i). \tag{A 18}$$

This is the step by which the pressure double arrows enter a diagram, and to give neater diagrams this step is omitted. From there on hydrodynamic couplings D are the only ones relevant to us since in the approximation we are considering, each pressure component enters each subdiagram only once.

Thus we first have in all contributions to $Z^{(n-1)}(\alpha_i, \alpha_i')$ the factor

$$\Pi^{(n)} = \prod_{i=1}^{n} \frac{\sigma_i^2}{\sigma_i^2 - \Omega_i^2}(n-1)! \tag{A 19}$$

Further, we have the contribution given by the n-point coupling constant $D(\alpha_i; (\alpha_1, \ldots, \alpha_n), {}_i')$. (This contribution occurs $(n-1)!$ times because of the symmetry in the last indices. The factor $(n-1)!$ is included in Π.) The factor $\Gamma^{(n-1)}$ is then obtained from the diagram as the remaining contribution with $n-1$ lines entering, so that with $P_\gamma = -\sigma_\gamma^2/(\omega_\gamma^2 - \sigma_\gamma^2)$

$$\Gamma^{(n-1)}(\alpha_i; (\alpha)_i')(n-1)! = \sum_{\beta_i} \{G^{(n-2)}(\alpha_i; \beta_1 \ldots \beta_{n-3}, \gamma)\, P_\gamma\, G^{(2)}(-\gamma; \beta_{n-2}, \beta_{n-1})$$
$$+ G^{(n-3)}(\alpha_i; \beta, \gamma)\, P_\gamma\, G^{(3)}(-\gamma; \beta) + \ldots + G^{(2)}(\alpha_i; \beta_1, \gamma)\, P_\gamma\, G^{(n-2)}(-\gamma; \beta_2 \ldots \beta_{n-1})\}. \tag{A 20}$$

Here the β_i are to be summed over all permutations of $(\alpha_i, \ldots, \alpha_n)'_i$, and γ is chosen such that in $G^{(n-j)}$ we have $(\alpha_i + \gamma + \beta_1 \ldots \beta_{n-j-1}) = 0$.

Since the functions $G^{(n-j)}$ are symmetric functions of their arguments (inductive assumption), $\Gamma^{(n-1)}$ is also a symmetric function of its arguments. To see this, consider for instance $\alpha_i = \alpha_2$ and interchange α_1 and α_2. In all contributions we either: (i) change a term $G^{(n-j)}(\alpha_2; \alpha_1 \ldots)$ to $G^{(n-j)}(\alpha_1; \alpha_2 \ldots)$, which alters nothing because of the symmetry of $G^{(n-j)}$; or (ii) change $G^{(n-j)}(\alpha_2 \ldots \gamma)\, G^{(j)}(-\gamma; \alpha_1 \ldots)$ to $G^{(n-j)}(\gamma'; \ldots \alpha_1 \ldots) \times G^{(j)}(\alpha_2; \ldots -\gamma')$, which is a term also contained in the sum $\Gamma^{(n-1)}(\alpha_2; \ldots)$. (We either (i) interchange lines within a blob or (ii) interchange blobs.)

Thus everything has now been proved, except for step 2. We shall now omit the index (n). We now have the information

$$\Sigma a_i D(\alpha_i, (\alpha_1, \ldots, \alpha_n)'_i) \equiv 0 \qquad \text{(A 21)}$$

if

$$\Sigma \alpha_i = 0$$

and the additional constraints (A 12) are satisfied. Since

$$\sum_i \alpha_i = 0$$

we may rewrite (A 21) as

$$\sum_{i=2}^{n} \alpha_i D_i(\alpha_2, \ldots, \alpha_n) \equiv 0, \qquad \text{(A 22)}$$

with

$$D_i(\alpha_2, \ldots, \alpha_n) = D(\alpha_i; (\alpha_1, \ldots, \alpha_n)'_i) - D(\alpha_1; \alpha_2, \ldots, \alpha_n), \qquad \text{(A 23)}$$

the prime in $(\alpha_1, \ldots, \alpha_n)'_i$ indicating that i is missing in the argument list. Our aim is then to show that

$$D_i(\alpha_2, \ldots, \alpha_n) \equiv 0. \qquad \text{(A 24)}$$

For $n \leqslant 4$ the proof is simple, because except for certain configurations of the α_i, which can be filled in by continuation, the determinant $|\boldsymbol{\alpha}_2 \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_4| \neq 0$, so that D_i must vanish. For $n > 4$ we unfortunately have nothing simple to offer. (The author is convinced that by using the known symmetries of D in the last indices and the invariance under reflexions and rotations of the $\mathbf{k}_i$ a more elegant proof than the following monstrosity would be possible.)

Consider the equations of motion for surface waves:

$$\dot{\zeta} - \phi_{,3} + \zeta_{,i} \phi_{,i} = 0 \quad (z = \zeta), \qquad \text{(A 25)}$$

$$\dot{\phi} + g\zeta + \tfrac{1}{2}\phi_{,i}\phi_{,i} = -\rho^{-1} p^{(a)} \quad (z = \zeta), \qquad \text{(A 26)}$$

$$\Delta\phi = 0, \quad \phi \to 0 \quad \text{for} \quad z \to -\infty. \qquad \text{(A 27)}$$

We indicate how we would construct the D's in order to deduce some properties. First we obtain $\Phi^{\omega}_{\mathbf{k}}$ $(z = 0)$ from (A 25):

$$k\phi^{\omega}_{\mathbf{k}}(z = 0) + \sum_{n=2}^{\infty} \tilde{A}^{-\omega\omega_1 \ldots \omega_n}_{-\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_n} \Phi^{\omega_1}_{\mathbf{k}_1}(z = 0)\, \zeta^{\omega_1}_{\mathbf{k}_1} \ldots \zeta^{\omega_n}_{\mathbf{k}_n} = -i\omega\zeta^{\omega}_{\mathbf{k}}. \qquad \text{(A 28)}$$

The $\tilde{A}$'s are uniquely determined by (A 25) and (A 27). Equation (A 25) can be solved iteratively:

$$k\phi^{\omega}_{\mathbf{k}}(z = 0) = -i\omega\zeta^{\omega}_{\mathbf{k}} + \sum_{n=2}^{\infty} \hat{A}^{-\omega\omega_1 \ldots \omega_n}_{-\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_n} \zeta^{\omega_1}_{\mathbf{k}_1} \ldots \zeta^{\omega_n}_{\mathbf{k}_n}, \qquad \text{(A 29)}$$

$\hat{A}$ being linear in the ω_i. This can then again be inserted into (A 26), using again an expansion about $z = 0$ and thus we finally obtain (A 1); we wish to stress only that for any n the D's are homogeneous quadratic forms in the ω_i.

Thus in (A 22) we have

$$D_i(\alpha_2, \ldots, \alpha_n) = \sum_{j,k} H_{ijk}\,\omega_j\,\omega_k \tag{A 30}$$

with $$H_{ijk} = H_{ikj}, \quad H_{ijk} = H_{ijk}(\mathbf{k}_2, \ldots, \mathbf{k}_n). \tag{A 31}$$

Thus considering the ω component of (A 22) only we have for all ω_i

$$\sum_{i,j,k=2}^{n} H_{ijk}\,\omega_i\,\omega_j\,\omega_k \equiv 0 \tag{A 32}$$

and the coefficients of $\omega_i\,\omega_j\,\omega_k$ must vanish. Consider, for example, $(i,j,k) = (2,3,4)$, then

$$H_{234} + H_{324} + H_{423} \equiv 0. \tag{A 33}$$

Further we have

$$2H_{223} + H_{322} \equiv 0 \tag{A 34}$$

and $$H_{iii} \equiv 0, \quad i = (2, \ldots, n). \tag{A 35}$$

Now it is sufficient to prove that in addition to (A 35)

$$H_{233} \equiv H_{223} \equiv H_{234} \equiv 0 \tag{A 36}$$

because of the symmetries of the H_i under permutations. To see this, observe that the symmetry of the D coupling in the last $n-1$ indices, and the definition (A 23) of the D_i, implies that

$$H_{ijk}(\mathbf{k}_2, \ldots, \mathbf{k}_n) = H_{\mu_i\,\mu_j\,\mu_k}(\mathbf{k}_{\nu_2}, \ldots, \mathbf{k}_{\nu_n}). \tag{A 37}$$

Here $\nu\colon i \rightarrow \nu_i$ is any permutation of $(2, \ldots, n)$ and $\mu\colon i \rightarrow \mu_i$ is the inverse permutation; $\mu = \nu^{-1}$.

Further from (A 23) we have

$$D_2(\alpha_2, \ldots, \alpha_n) = -D_2(\bar{\alpha}_2, \alpha_3, \ldots, \alpha_n) \tag{A 38}$$

with $\bar{\alpha}_2 = -(\alpha_2 + \ldots + \alpha_n) = \alpha_1$. We also have

$$D_3(\alpha_1, \alpha_3, \ldots, \alpha_n) = D_2(\alpha_3, \alpha_2, \ldots, \alpha_n) - D_2(\alpha_2, \alpha_3, \ldots, \alpha_n) \tag{A 39}$$

and $$D_3(\alpha_1, \alpha_3, \ldots, \alpha_n) = D_2(\alpha_3, \alpha_1, \ldots, \alpha_n). \tag{A 40}$$

Inserting (A 30) and $\omega_1 = -(\omega_2 + \ldots + \omega_n)$ and again using $H_{iii} \equiv 0$, (A 39) and (A 40) yield, writing $\mathbf{e} = (\mathbf{k}_4, \ldots, \mathbf{k}_n)$,

$$-H_{223}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{e}) + H_{223}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{e}) - H_{223}(\mathbf{k}_3, \mathbf{k}_1, \mathbf{e}) \equiv 0. \tag{A 41}$$

Now returning to (A 38), we note that we obtain three additional equations ($\mathbf{k}_1 \equiv \bar{\mathbf{k}}_2$):

$$H_{223}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{e}) = H_{223}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{e}), \tag{A 42}$$

$$H_{233}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{e}) = -H_{233}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{e}) + 2H_{223}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{e}), \tag{A 43}$$

$$H_{234}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{e}) = -H_{234}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{e}) + \tfrac{1}{2}(H_{223} + H_{224})\,(\mathbf{k}_1, \mathbf{k}_2, \mathbf{e}). \tag{A 44}$$

We first show that $H_{223} \equiv 0$. With (A 34) and (A 37) this also proves that $H_{233} \equiv 0$. Equations (A 34) and (A 37) yield

$$2H_{223}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{e}) = -H_{233}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{e}), \tag{A 45}$$

so that (A 43) yields

$$H_{223}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{e}) + H_{223}(\mathbf{k}_3, \mathbf{k}_1, \mathbf{e}) + H_{223}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{e}) \equiv 0, \tag{A 46}$$

and with (A 42) this may be written as

$$H_{223}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{e}) + H_{223}(\mathbf{k}_3, \mathbf{k}_1, \mathbf{e}) + H_{223}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{e}) \equiv 0. \tag{A 47}$$

The sum (A 47) + (A 41) then yields

$$H_{223}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{e}) + H_{223}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{e}) \equiv 0. \tag{A 48}$$

Thus whenever $\mathbf{k}_2 = \mathbf{k}_3$, H_{223} vanishes. But (A 42) then shows that H_{223} always vanishes, because we can reach any vector $\mathbf{k}_1$ by varying the set $\mathbf{e}$.

It remains to be shown that $H_{234} \equiv 0$. Introducing $\mathbf{e}' = (\mathbf{k}_5, \ldots, \mathbf{k}_n)$, we may write (A 36) as

$$H_{234}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4, \mathbf{e}') + H_{234}(\mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_2, \mathbf{e}') + H_{234}(\mathbf{k}_4, \mathbf{k}_3, \mathbf{k}_3, \mathbf{e}') \equiv 0. \tag{A 49}$$

If $\mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4$, $H_{234} = 0$. From (A 44) we see then that also

$$H_{234}(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_3, \mathbf{e}') \equiv 0. \tag{A 50}$$

Thus from (A 49), observing that $H_{234}(a, b, c) = H_{234}(a, c, b)$,

$$H_{234}(\mathbf{k}_3, \mathbf{k}_3, \mathbf{e}) \equiv 0, \quad \mathbf{e} = (\mathbf{k}_4, \ldots, \mathbf{k}_n), \quad \text{arbitrary}. \tag{A 51}$$

Again with (A 44) we conclude

$$H_{234}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{e}) \equiv 0 \tag{A 52}$$

or

$$H_{234} \equiv 0. \tag{A 53}$$

This finally concludes the proof.

We close with a few remarks on the constraint removal problem. This is obviously a purely technical problem, even though it may be complicated mathematically. Whenever we encounter a situation where a subset satisfies $\Sigma\alpha_i = 0$, we have balancing of input at a lower level and a corresponding singularity of the coupling. These singularities are removed by considering the cumulants of the forcing pressure field. The other constraint, that we should not couple into free waves, is easier to remove and clearly corresponds to higher-order forcing of free waves, so that input balances growth.

The relevance of the extended symmetry

We gain a bit more insight into the necessity for introducing off-shell coupling as soon as we have external forcing through a short sketch of the Lagrangian formulation. We shall also see that the extended symmetry contains new knowledge, and is not just a clumsy presentation of old results. Following Whitham (1966) we introduce the Lagrangian $L = \int \tilde{L}\, dx\, dt$:

$$\tilde{L} = \int_{-\infty}^{\zeta} (\phi_t + \tfrac{1}{2}\phi_{,i}\phi_{,i} + gz)\, dz + \zeta p^{(a)}(t)\frac{1}{\rho}. \tag{A 54}$$

For simplicity L will be varied with respect to ζ and those Φ's that already satisfy $\Delta\Phi = 0$.

Introducing $\zeta_k = r_k$, $q_k = \Phi_k(0)$ and $\dot{q}_k = \dot{\Phi}_k(0)$, we have

$$L = L(\dot{r}_k; q_k, \dot{q}_k). \tag{A 55}$$

Further introducing

$$s_k = \partial L/\partial \dot{r}_k, \quad p_k = \partial L/\partial \dot{q}_k, \tag{A 56}$$

we obtain the equations of motion via

$$H(p, s, q, p^{(a)}(t)) = p_k \dot{q}_k + s_k \dot{r}_k - L \tag{A 57}$$

as

$$\dot{q}_k = \partial H(p, q, s, p^{(a)})/\partial p_k, \quad \dot{r}_k = \partial H/\partial s_k, \qquad \text{(A 58), (A 59)}$$

$$\dot{p}_k = -\partial H/\partial q_k, \quad \dot{s}_k = -\partial H/\partial r_k = 0. \qquad \text{(A 60), (A 61)}$$

For the case $p^{(a)} \equiv 0$ we have $s_k = 0$, since $\partial L/\partial \dot{r}_k = 0$ yields the pressure condition

$$\phi_t + \tfrac{1}{2}\phi_{,i}\phi_{,i} + gz = 0 \quad \text{on} \quad z = \zeta. \tag{A 62}$$

Now if for zero atmospheric pressure

$$\partial L(p^{(a)} = 0)/\partial \dot{r}_k = f_0(\dot{r}, \dot{q}_k, q_k) \tag{A 63}$$

and (A 56) has the inversion for $(p^{(a)} = 0)$

$$\dot{r}_k = g_0(p_k, q_k, s_k), \quad \dot{q}_k = h_0(p_k, q_k, s_k) \tag{A 64}$$

[incidentally defining the C coefficients of (A 5) and (A 6)], we see that if $p^{(a)} \neq 0$ we have

$$\left.\begin{aligned} \dot{r}_k &= g_0(p_k, q_k, s_k - \rho^{-1}p^{(a)}_{-k}(t)), \\ \dot{q}_k &= h_0(p_k, q_k, s_k - \rho^{-1}p^{(a)}_{-k}(t)). \end{aligned}\right\} \tag{A 65}$$

Inspection of (A 57) then shows that the equations of motion, which in the zero-pressure case reduced to solving (we denote the variable s_k by σ_k in H_0)

$$\left.\begin{aligned} \dot{q}_k &= \partial H_0(p_k, q_k, \sigma_k = 0)/\partial p_k, \\ \dot{p}_k &= -\partial H_0(p_k, q_k, \sigma_k = 0)/\partial q_k \end{aligned}\right\} \tag{A 66}$$

(which can trivially be symmetrized by transforming to $a^{(\pm)} = p \mp iq$), now reduce to, since we again have $s_k = 0$,

$$\left.\begin{aligned} \dot{q}_k &= \partial H_0(p_k, q_k, \sigma_k = -\rho^{-1}p^{(a)}_{-k}(t))/\partial p_k + \gamma_1, \\ \dot{p}_k &= -\partial H_0(p_k, q_k, \sigma_k = -\rho^{-1}p^{(a)}_{-k}(t))/\partial q_k + \gamma_2, \end{aligned}\right\} \tag{A 67}$$

γ_1 and γ_2 arising from the contribution $-L = -L_0 - \dot{r}p^{(a)}$ in (A 57). Thus in order to solve (A 67) we need to know the function H_0 not only on $\sigma = 0$, but also for $\sigma \neq 0$. This presumably corresponds to knowledge of off-shell coupling coefficients. This aspect has not been studied in detail, but it seems that the knowledge of the coupling coefficients $D(\alpha_1, \ldots, \alpha_n)$ at all those points which can be constructed by adding and subtracting the 'free' $\beta_i = (\pm\sigma(k_i), k_i)$ is equivalent to complete information about the function $H_0(p, q, s = 0)$ and additionally the structure of the $\hat{C}(k_i, \nu_i)$ in (A 6). The information D contains at the remaining points $(\alpha_1, \ldots, \alpha_n)$ should be equivalent to information on the functions g_0 and h_0 for $s \neq 0$ and information about $H_0(p, q, s \neq 0)$.

REFERENCES

Dobson, F. W. 1971 Measurements of atmospheric pressure on wind generated sea waves. *J. Fluid Mech.* **48**, 91–127.

Elliott, J. A. 1972 Microscale pressure fluctuations near waves being generated by wind. *J. Fluid Mech.* **54**, 427–448.

Garrett, C. & Smith, J. 1976 On the interaction between long and short surface waves. *J. Phys. Ocean.* **6**, 925.

Hasselmann, D. E. 1976 *Sonderforschungsbereich Meeresforschung*, SFB 94, *Universität Hamburg, Rep.* no. 10.

Hasselmann, K. 1966*a* Feynman diagrams and interaction rules of wave–wave scattering processes. *Rev. Geophys.* **4**, 1–32.

Hasselmann, K. 1966*b* Nonlinear interactions treated by the methods of theoretical physics. *Proc. Roy. Soc.* A **299**, 77–100.

Hasselmann, K. 1968 Weak interaction theory of ocean waves. In *Basic Developments in Fluid Dynamics*, vol. 2, pp. 117–182. Academic Press.

Hasselmann, K. *et al.* 1973. Measurements of wind wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP). *Ergänzungsheft Dtsch. Hydrograph. Z. Reihe* A (8°), no. 12.

Longuet-Higgins, M. S. 1969 On the action of a variable stress at the surface of water waves. *Phys. Fluids* **12**, 737–740.

Miles, J. W. 1957 On the generation of surface waves by shear flows. *J. Fluid Mech.* **3**, 185.

Newell, A. C. 1968 The closure problem in a system of random gravity waves. *Rev. Geophys.* **6**, 1–31.

Snyder, R. L. 1974 A field study of wave-induced pressure fluctuations above surface gravity waves. *J. Mar. Res.* **32**, 497–531.

Snyder, R. L. & Cox, C. S. 1966 A field study of the wind generation of ocean waves. *J. Mar. Res.* **24**, 141–178.

Valenzuela, G. R. & Wright, J. W. 1976 The growth of waves by modulated wind stress. *J. Geophys. Res.* **81**, 5794.

Whitham, G. B. 1966 Variational methods and applications to water waves. *Proc. Roy. Soc.* A **299**, 6–25.